Advanced Calculus

Limit of Functions and Continuity

ThinkBS: Basic Sciences in Engineering Education

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Assume that $f : (X, d_X) \to (Y, d_Y)$ is a function between two metric spaces which is defined in a deleted neighborhood $N_r^*(a)$ of $a \in X$.

We say that the limit of $f(x)$ as x tends to a is equal to L iff for any $\varepsilon > 0$ there exist a $\delta_\varepsilon > 0$ such that for every x, if $0 < d_X(x, a) < \delta_\varepsilon$ then $d_Y(f(x), L) < \varepsilon$.

In this case we write $\lim_{x \to a} f(x) = L$.

The definition above is equivalent to saying that:

$\lim_{x \to a} f(x) = L$ iff for any $\varepsilon > 0$ there exist a $\delta_\varepsilon > 0$ such that

$$N^*_{\delta_\varepsilon}(x) \subseteq f^{-1}(N_\varepsilon(L))$$

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Continuity of Functions

Assume that $f : (X, d_X) \to (Y, d_Y)$ is a function between two metric spaces.

We say that $f(x)$ is continuous at a iff for any $\varepsilon > 0$ there exist a $\delta_\varepsilon > 0$ such that for every x, if $d_X(x, a) < \delta_\varepsilon$ then $d_Y(f(x), f(a)) < \varepsilon$.

This is to say that f is defined in a and $\lim_{x \to a} f(x) = f(a)$.

The definition above is equivalent to saying that:

$\lim_{x \to a} f(x) = f(a)$ iff for any $\varepsilon > 0$ there exist a $\delta_\varepsilon > 0$ such that

$$N_{\delta_\varepsilon}(a) \subseteq f^{-1}(N_\varepsilon(f(a)))$$

If a function f is continuous on all points of a set A, then we say that f is continuous on A.

Other Formulations for Continuity of Functions

- A function $f : (X, d_X) \to (Y, d_Y)$, is continuous at point $a \in X$ iff for any sequence $\{a_n\} \subseteq X$, if $a_n \to a$, then $f(a_n) \to f(a)$.
- A function $f : (X, d_X) \to (Y, d_Y)$ is continuous if for any open set $O \subseteq Y$, $f^{-1}(O)$ is open in X.
- A function $f : (X, d_X) \to (Y, d_Y)$ is continuous if for any close set $C \subseteq Y$, $f^{-1}(C)$ is close in X.

For further information, see 'Chap. 4: Continuous Functions'.

Uniform Continuity

A function $f : (X, d_X) \to (Y, d_Y)$, is said to be uniformly continuous on a set $A \subseteq X$ iff for any $\varepsilon > 0$ there exist a $\delta_\varepsilon > 0$ such that for every x,$y \in X$ if $d_X(x, y) < \delta_\varepsilon$ then $d_Y(f(x), f(y)) < \varepsilon$.

Example: Any linear function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ for $a, b \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Every uniformly continuous function is continuous, but the converse is not true. For instance $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous on $\mathbb{R}$ but is not uniformly continuous on $\mathbb{R}$. (why?)