Advanced Calculus

Cauchy Sequences, Complete Spaces and Limits

ThinkBS: Basic Sciences in Engineering Education

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Cauchy Sequences and Complete spaces

Assume (X, d) is a metric space. We say that the sequence $\{a_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$ we have $d(a_n, a_m) < \varepsilon$.

Loosely speaking, a Cauchy sequence is a sequence for which the elements get closer and closer **all together** to each other after some step on.

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Example 2: Every Cauchy sequence in a metric space is bounded.

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A metric space (X, d) is called **complete** if every Cauchy sequence is convergent to a point of space.

$(\mathbb{Q}, |.|)$ is not a complete metric space but $(\mathbb{R}, |.|)$ is. (why?)

For further properties of Cauchy sequences, see 'Chap. 3: Cauchy Sequences'.

Assume that (X, d) is a metric space and $A \subseteq X$. In this case:

- $x \in \bar{A}$ iff there is a sequence $\{a_n\}$ in A such that $a_n \to x$.
- $x \in A'$ iff there is a sequence $\{a_n\}$ with distinct elements in A such that $a_n \to x$.
- $x \in A'$ iff there is a sequence $\{a_n\}$ in $A \setminus \{x\}$ such that $a_n \to x$.

This clarifies why we used 'cluster' and 'limit' points when we gave the definitions. Here one should consider the difference between a 'set-wise' point and 'sequence-wise' point.