$\dot{x} = Ax$ $\quad$ $n \times n$ real

$\lambda_1, \lambda_2, \ldots, \lambda_n$

$\mathrm{Re}(\lambda) < 0 \longrightarrow E^s$

$\mathrm{Re}(\lambda) = 0 \longrightarrow E^c$

$\mathrm{Re}(\lambda) > 0 \longrightarrow E^u$

THEOREM 4.49. *Consider a system*

$$\dot{x} = Ax, \tag{4.74}$$

where A is a real $n \times n$ matrix. Let E^s, E^u and E^c be the stable, unstable and center subspaces of the system. Then each of these three spaces are invariant with respect to the flow of system (4.74). Moreover, we have

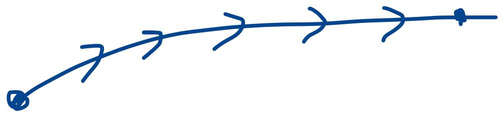
$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c. \tag{4.75}$$

$e^{At} x = e^{At} x^s + \cancel{e} e^{At} x^u + e^{At} x^c$

Proof. See [Per01]. $x \in \mathbb{R}^n$ $x = x^s + x^u + x^c$ □

PROPOSITION 4.50. [11] *Let O be the origin of $\mathbb{R}^n$. Consider a point $x_0 \in \mathbb{R}^n$, and let E^s, E^u and E^c be the stable, unstable and center subspaces of system (4.74), respectively. Then, the following hold.*

(i) If $x_0 \in E^s$, then $e^{tA}x_0 \to O$ as $t \to \infty$.

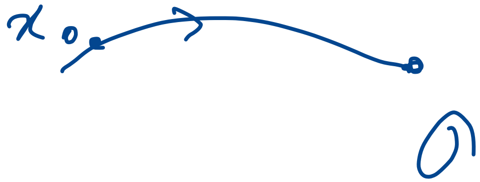


(i) If $x_0 \in E^u$, then $e^{tA}x_0 \to O$ as $t \to -\infty$.

(i) If $e^{tA}x_0 \to O$ as $t \to \infty$, then $x_0 \in E^s \oplus E^c$.

(i) If $e^{tA}x_0 \to O$ as $t \to -\infty$, then $x_0 \in E^u \oplus E^c$.

Proof. See [Per01]. □



[11] A more comprehensive version of this result is valid: E^s (resp. E^u) is indeed the set of all points x_0 in $\mathbb{R}^n$ that converge to O exponentially fast as $t \to \infty$ (resp. $t \to -\infty$), i.e. $\exists a > 0$, $M > 0$ such that $\|e^{tA}x_0\| \leq Me^{-a|t|}$ for $t \geq 0$ (resp. $t \leq 0$). On the other hand, E^c is the set of the points $x_0 \in \mathbb{R}^n$ whose orbits grow at most sub-exponentially fast as $t \to \pm\infty$, i.e. $\forall a > 0$, $\frac{\|e^{tA}x_0\|}{e^{a|t|}} \to 0$ as $t \to \pm\infty$ (see [Rob98], Theorem 6.1).

Definition 4.51. *Let O be the origin of $\mathbb{R}^n$ and consider system (4.74).* $\mathbb{R}^n = E^s \oplus E^u$

(i) *We say the equilibrium O is hyperbolic if A has no eigenvalue with zero real part, i.e. $E^c = \{O\}$, or equivalently $\mathbb{R}^n = E^s \oplus E^u$. Otherwise, we say O is nonhyperbolic, i.e. A has at least one eigenvalue with zero real part.*

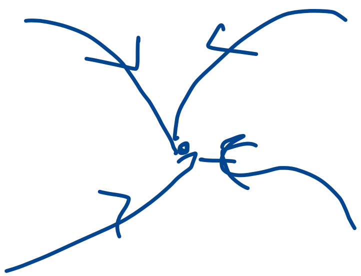
(ii) *We say the equilibrium O is a sink (resp. source) if all the eigenvalues of A have negative (resp. positive) real parts, i.e. $E^s = \mathbb{R}^n$ (resp. $E^u = \mathbb{R}^n$).*

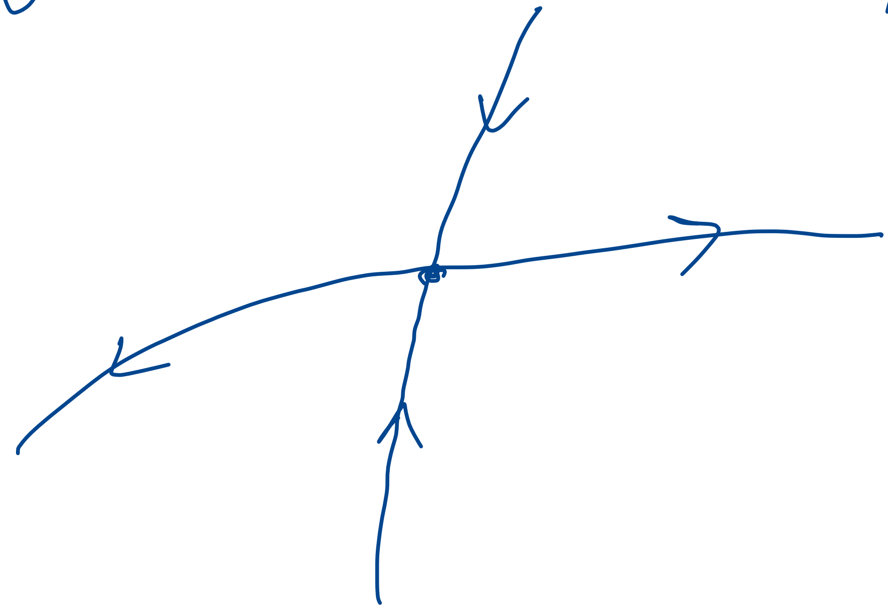
(iii) *We say the equilibrium O is a saddle if it is hyperbolic, and the matrix A has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part, i.e. $\mathbb{R}^n = E^s \oplus E^u$, $\dim(E^s) \geq 1$ and $\dim(E^u) \geq 1$.*

$\dot{x} = Ax$, O

$\mathbb{R}^n = E^s \oplus E^u$

$\mathbb{R}^n = E^s$





4.6. **An example of synchronization in linear systems**

Recall from the first lecture that a mathematical model for understanding synchronization in networks is given by

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \dots, N\}, \tag{4.76}$$

where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2(\mathbb{R}^n)$.

We now discuss a simple case of this model: two identical linear systems defined on $\mathbb{R}$ that are linearly coupled together. First, consider two identical linear systems

$$\dot{x}_1 = ax_1 \tag{4.77}$$

$a \neq 0$

and

$$\dot{x}_2 = ax_2, \tag{4.78}$$

where $x_i \in \mathbb{R}$ $(i = 1, 2)$ and a is a non-zero real constant. The dynamics of these systems are simple: for any initial point $x_i(0)$, we have $x_i(t) = e^{at}x_i(0)$, $i = 1, 2$. Thus, $x_i(t) \to 0$ if $a < 0$, and $|x_i(t)| \to \infty$ if $a > 0$.

(a) Case $a < 0$. (b) Case $a > 0$.

Figure 27: Phase portrait of $\dot{x}_i = ax_i$, where $i = 1, 2$.

We can couple these two systems as in (4.76). We have

$$\begin{aligned} \dot{x}_1 &= ax_1 + \alpha\,(x_2 - x_1) \\ \dot{x}_2 &= ax_2 + \alpha\,(x_1 - x_2) \end{aligned} \tag{4.79}$$

where α, the coupling strength, is a real constant. In terms of the notations in (4.76), we are considering $n = 1$, $N = 2$, $f_1(x) = f_2(x) = ax$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $H_i =$ identity.

We say system (4.79) gets into (complete) synchrony if for any initial condition $(x_1(0), x_2(0)) \in \mathbb{R}^2$, we have

$$\lim_{t\to\infty} |x_1(t) - x_2(t)| = 0. \tag{4.80}$$

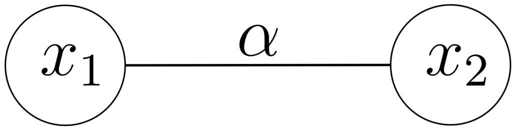


Figure 28: Two coupled systems.

x_{10}

x_{20}

- Define $z(t) := x_1(t) - x_2(t)$. To detect the synchrony in system (4.79), we need to see when $z(t) \to 0$ as $t \to \infty$.
- Recall that

$$\begin{cases} \dot{x}_1 = (a - \alpha)\, x_1 + \alpha x_2 \\ \dot{x}_2 = \alpha x_1 + (a - \alpha)\, x_2 \end{cases} \tag{4.81}$$

- We have

$z \in \mathbb{R}$

$$\dot{z} = \dot{x}_1 - \dot{x}_2 = (a - 2\alpha)\, z. \tag{4.82}$$

This yields $z\,(t) = e^{(a-2\alpha)t} z\,(0)$. Thus, $\lim_{t\to\infty} z(t) = 0$ for arbitrary $z(0)$ if and only if $a - 2\alpha < 0$.

- Define $\alpha_c := \frac{a}{2}$. This constant is called the critical coupling value. We have that system (4.79) gets into (complete) synchrony if and only if $\alpha > \alpha_c$.

5. Nonlinear systems

$f(x) = Ax$

5.1. Stability of equilibria

Consider a system C

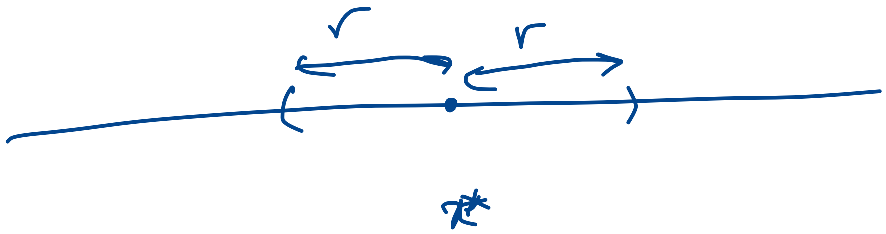
$$\boxed{\dot{x} = f(x),} \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Assume that $x^* \in \mathbb{R}^n$ is an equilibrium of this system, i.e. $f(x^*) = 0$. Let $r > 0$. An open ball in $\mathbb{R}^n$ with radius r, centered at x^* is defined by

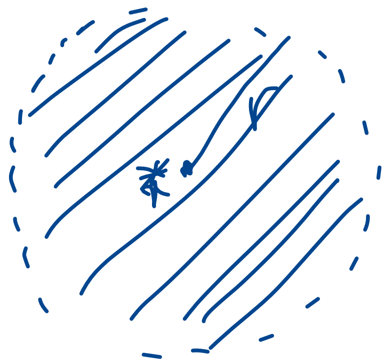
$$B_r(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| < r\}. \tag{5.2}$$

$f(x^*) = 0$

$\mathbb{R}$



x^*



For arbitrary $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0) = \phi(t, x_0)$ be the solution of (5.1) with the initial condition $\phi(0, x_0) = x_0$.

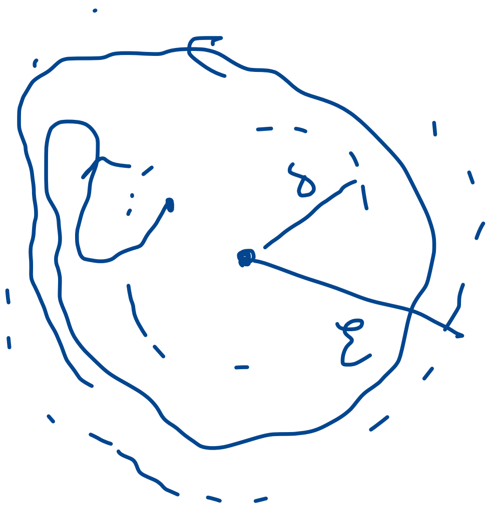
Definition 5.1. *The equilibrium x^* is called stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$ and for all $t \geq 0$, we have $\phi_t(x_0) \in B_\epsilon(x^*)$.*

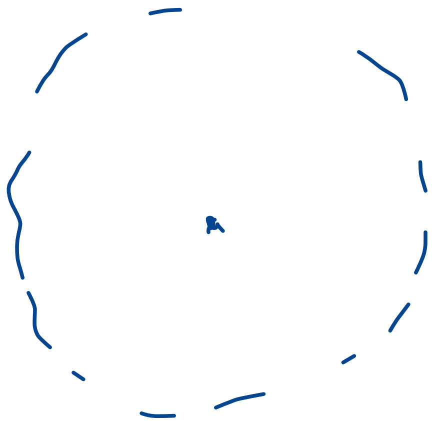
Definition 5.2. *The equilibrium x^* is called unstable if it is not stable.*

Definition 5.3. *The equilibrium x^* is called asymptotically stable if it is stable, and there exists a $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$, we have $\lim_{t\to\infty} \phi_t(x_0) = x^*$.*

$x = f(x)$

x^*





$\delta = \varepsilon$

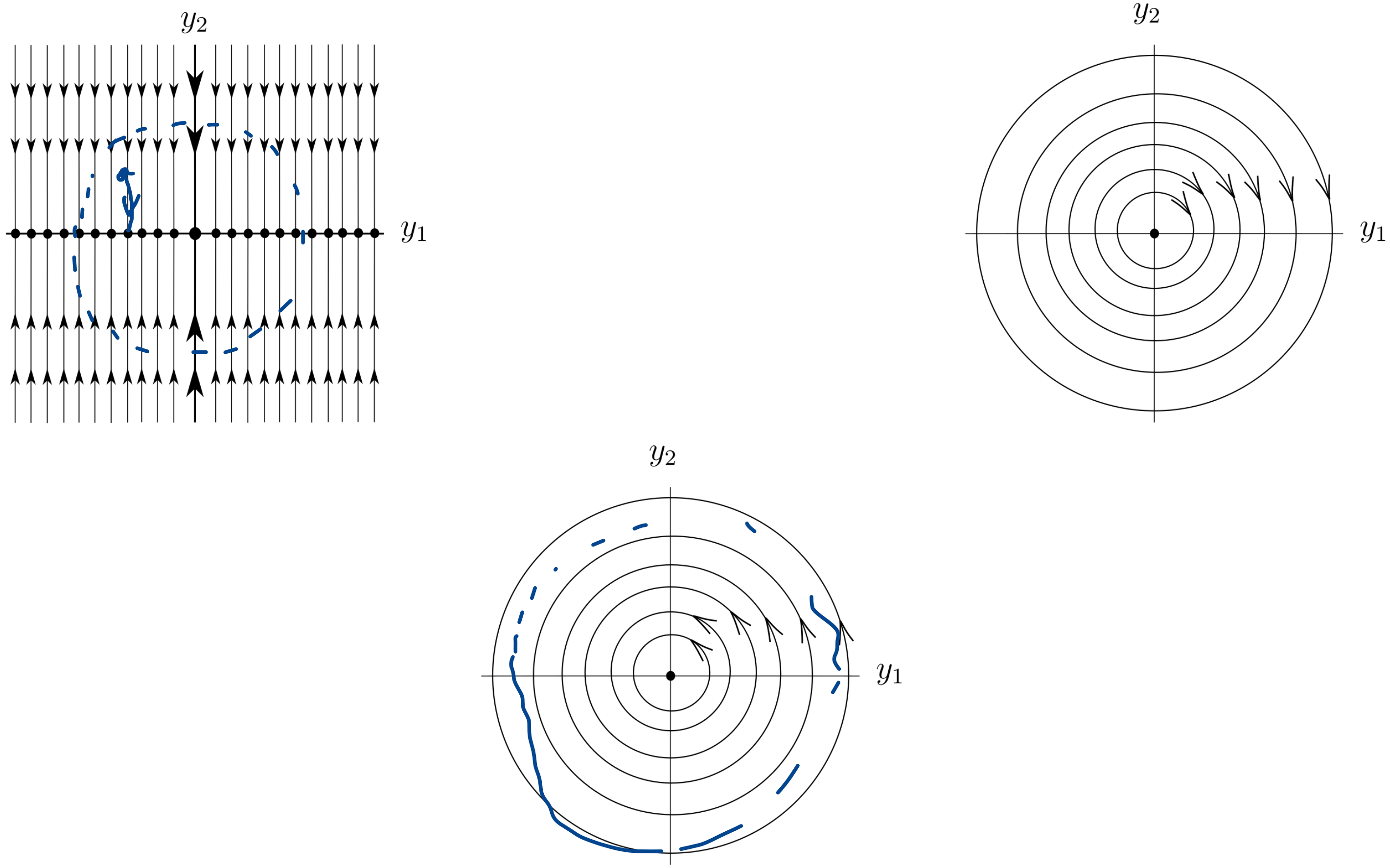


Figure 29: The origin is stable but not asymptotically stable.

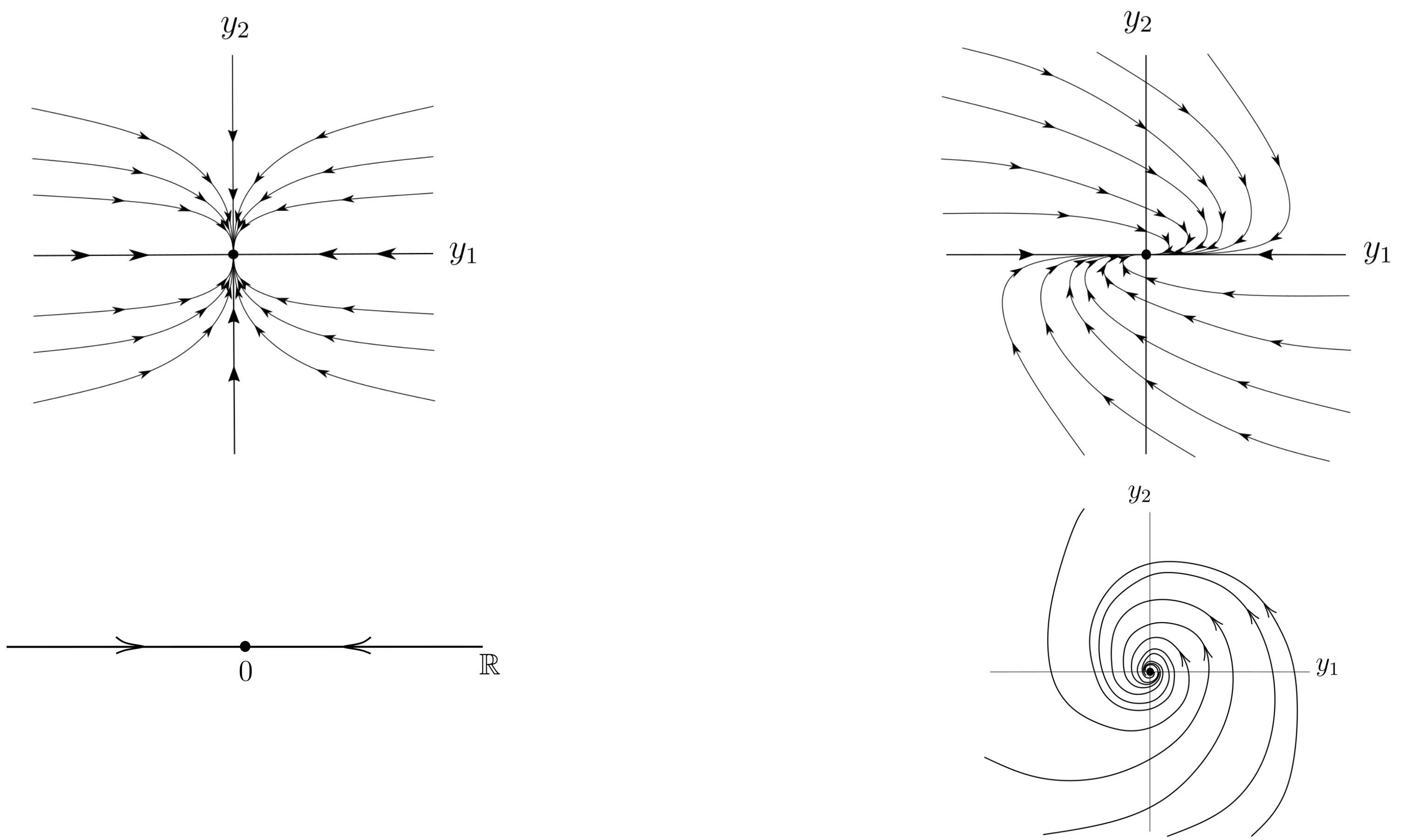


Figure 30: The origin is asymptotically stable.

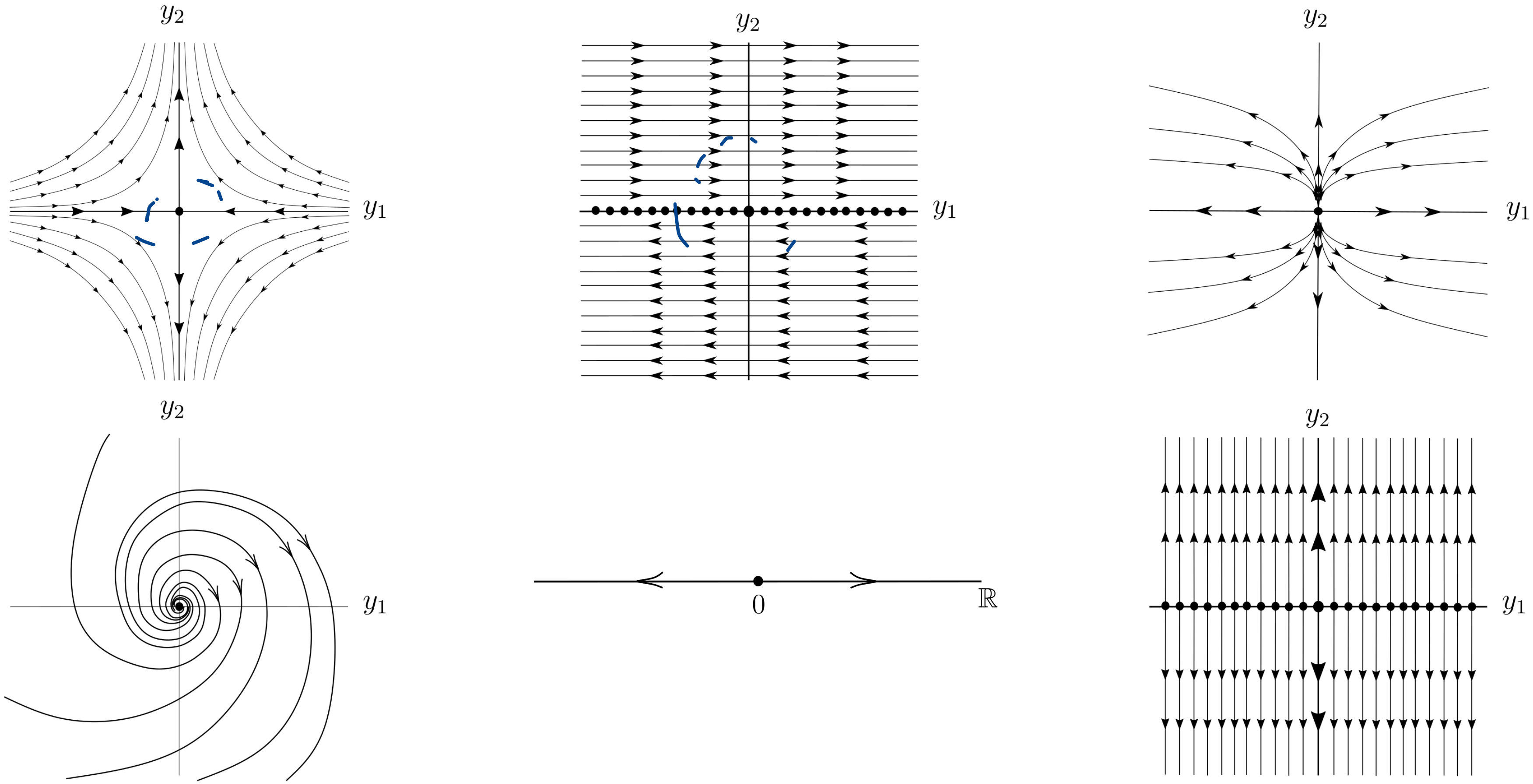


Figure 31: The origin is an unstable equilibrium.

Consider a system

$$\dot{x} = f(x), \tag{5.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Assume that the origin of $\in \mathbb{R}^n$ is an equilibrium of this system[12], i.e. $f(0) = 0$.

- **Question**: what is the stability of the equilibrium point at the origin? Stable? Asymptotically stable? Or unstable?

- **Answer to the question for the linear case:**
  Consider the case that f is linear, i.e. $f(x) = Ax$, for some $A \in \mathbb{R}^{n\times n}$. Indeed,

$$\dot{x} = Ax. \tag{5.4}$$

PROPOSITION 5.4. *Consider the decomposition $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ for system (5.4). Then*

(i) If the origin is stable, then $E^u = \{0\}$. In other words, if $E^u \neq \{0\}$, then the origin is unstable.

(ii) The origin is asymptotically stable if and only if $E^s = \mathbb{R}^n$, i.e. $E^u = E^c = \{0\}$.

Proof. See [Per01], Theorems 2 and 3 of Section 1.9 with their proofs. □

Remark 5.5. *In the case that $E^u = \{0\}$ and $E^c \neq \{0\}$, further investigation is needed to determine the stability of the equilibrium state at the origin (see e.g. [Per01], Problem 5 of the problem set of Section 1.9).*

[12] We do not lose generality by assuming that the equilibrium is located at the origin (see Remark 3.18).

5.2. **Linearization**

5.2.1 Preliminaries: Taylor's Theorem

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$-smooth function. We write

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1\left(x_1, x_2, \ldots, x_n\right) \\ \vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) \end{pmatrix}, \tag{5.5}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$ are $\mathcal{C}^1$.

THEOREM 5.6. *(Taylor's Theorem) Assume f is k-times continuously differentiable at the origin. Let α_i, $i = 1, \ldots, n$ be non-negative integers. Define $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then, for $|\alpha| = k$ and $i = 1, \ldots, n$, there exist functions $h_{i,\alpha}(x)$, such that $\lim_{x\to 0} h_{i,\alpha}(x) = 0$, and*

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f_1(0)\, x^\alpha + \sum_{|\alpha|=k} h_{1,\alpha}(x)\, x^\alpha \\ \vdots \\ \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f_n(0)\, x^\alpha + \sum_{|\alpha|=k} h_{n,\alpha}(x)\, x^\alpha \end{pmatrix}. \tag{5.6}$$

Remark 5.7. *Roughly speaking, Taylor's Theorem states that, close to the origin, we can write $f(x)$ as $P(x) +'$ remainder$'$, where P is a (non-zero) polynomial and the remainder is a function of x which often! can be neglected since, in comparison to $P(x)$, it is small.*

$f = P + R$

Example 5.8. (Taylor's Theorem for $n = 1$) Suppose $f : \mathbb{R} \to \mathbb{R}$ is k-times continuously differentiable. Then, we can write $f(x) = P(x) + R(x)$, for

$$P(x) = f(0) + \frac{1}{1!} f'(0)\, x + \frac{1}{2!} f''(0)\, x^2 + \frac{1}{3!} f'''(0)\, x^3 + \cdots + \frac{1}{k!} f^{(k)}(0)\, x^k, \tag{5.7}$$

and

$$R(x) = h(x)\, x^k \tag{5.8}$$

$h(x)\, x^{10}$

where $f^{(k)}$ stands for the k-th derivative, and $\lim_{x\to 0} h(x) = 0$.

$5x^2$

Example 5.9. Consider the function $f(x) = \sin x$. This function is $\mathcal{C}^\infty$ (we can differentiate it as many times as we want). Write $\sin x = P(x) + R(x)$, where P is a polynomial of degree k, and R satisfies $\lim_{x\to 0} \frac{R(x)}{x^k} = 0$. The following are some examples of P and R.

(i) $P(x) = x$, *and* $R(x) = h(x)\, x$ *such that* $\lim_{x\to 0} h(x) = 0$.

(ii) $P(x) = x - \frac{x^3}{3!}$, *and* $R(x) = h(x)\, x^3$ *such that* $\lim_{x\to 0} h(x) = 0$.

(iii) $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$, *and* $R(x) = h(x)\, x^{11}$ *such that* $\lim_{x\to 0} h(x) = 0$.

(iv) $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$, *and* $R(x) = h(x)\, x^{12}$ *such that* $\lim_{x\to 0} h(x) = 0$.

Example 5.10. *(Taylor's Theorem for $n=2$) Suppose $f:\mathbb{R}^2\to\mathbb{R}^2$, given by $x\mapsto(f_1(x),f_2(x))$, where $x=(x_1,x_2)$, is 3-times continuously differentiable. We can write*

$f(x_1, x_2) =$

$$\begin{aligned}
f_1(x_1,x_2) = f_1(0,0) \\
&+ \frac{1}{1!}\cdot\frac{\partial f_1}{\partial x_1}(0,0)\,x_1 + \frac{1}{1!}\cdot\frac{\partial f_1}{\partial x_2}(0,0)\,x_2 \\
&+ \frac{1}{2!}\cdot\frac{\partial f_1}{\partial x_1^2}(0,0)\,x_1^2 + \frac{1}{1!1!}\cdot\frac{\partial f_1}{\partial x_1\partial x_2}(0,0)\,x_1x_2 + \frac{1}{2!}\cdot\frac{\partial f_1}{\partial x_2^2}(0,0)\,x_2^2 \\
&+ \frac{1}{3!}\cdot\frac{\partial f_1}{\partial x_1^3}(0,0)\,x_1^3 + \frac{1}{2!1!}\cdot\frac{\partial f_1}{\partial x_1^2\partial x_2}(0,0)\,x_1^2x_2 + \frac{1}{1!2!}\cdot\frac{\partial f_1}{\partial x_1\partial x_2^2}(0,0)\,x_1x_2^2 + \frac{1}{3!}\cdot\frac{\partial f_1}{\partial x_2^3}(0,0)\,x_2^3 \\
&+ h_{1,30}(x)\,x_1^3 + h_{1,21}(x)\,x_1^2x_2 + h_{1,12}(x)\,x_1x_2^2 + h_{1,03}(x)\,x_2^3
\end{aligned} \tag{5.9}$$

such that all the functions h converge to 0 as $x\to 0$. Regarding f_2, we have

$$\begin{aligned}
f_2(x_1,x_2) = f_2(0,0) \\
&+ \frac{1}{1!}\cdot\frac{\partial f_2}{\partial x_1}(0,0)\,x_1 + \frac{1}{1!}\cdot\frac{\partial f_2}{\partial x_2}(0,0)\,x_2 \\
&+ \frac{1}{2!}\cdot\frac{\partial f_2}{\partial x_1^2}(0,0)\,x_1^2 + \frac{1}{1!1!}\cdot\frac{\partial f_2}{\partial x_1\partial x_2}(0,0)\,x_1x_2 + \frac{1}{2!}\cdot\frac{\partial f_2}{\partial x_2^2}(0,0)\,x_2^2 \\
&+ \frac{1}{3!}\cdot\frac{\partial f_2}{\partial x_1^3}(0,0)\,x_1^3 + \frac{1}{2!1!}\cdot\frac{\partial f_2}{\partial x_1^2\partial x_2}(0,0)\,x_1^2x_2 + \frac{1}{1!2!}\cdot\frac{\partial f_2}{\partial x_1\partial x_2^2}(0,0)\,x_1x_2^2 + \frac{1}{3!}\cdot\frac{\partial f_2}{\partial x_2^3}(0,0)\,x_2^3 \\
&+ h_{2,30}(x)\,x_1^3 + h_{2,21}(x)\,x_1^2x_2 + h_{2,12}(x)\,x_1x_2^2 + h_{2,03}(x)\,x_2^3
\end{aligned} \tag{5.10}$$

such that all the functions h converge to 0 as $x\to 0$.

5.2.2 Linearization at equilibria: stable and unstable invariant manifolds theorem

Consider a system

$$\dot{x} = f(x), \qquad = \cancel{f(0)}^{0} + Df(0)x + \overbrace{h(x)x} \tag{5.11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Think of f as $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$. Assume that the origin is an equilibrium of this system, i.e. $f(0) = 0$. According to Taylor's theorem, we can write system (5.11) as

$$\dot{x} = Df(0)\,x + F(x), \tag{5.12}$$

where $F(x) = h(x)\,x$, for some h satisfying $\lim_{x\to 0} h(x) = 0$, and $Df(0)$ (called the derivative or differential of f at 0) is given by the $n \times n$ matrix

$$Df(0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(0) & \frac{\partial f_1}{\partial x_2}(0) & \cdots & \frac{\partial f_1}{\partial x_n}(0) \\ \frac{\partial f_2}{\partial x_1}(0) & \frac{\partial f_2}{\partial x_2}(0) & \cdots & \frac{\partial f_2}{\partial x_n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(0) & \frac{\partial f_n}{\partial x_2}(0) & \cdots & \frac{\partial f_n}{\partial x_n}(0) \end{pmatrix}. \tag{5.13}$$

Definition 5.11. *Consider system (5.11). The expressions $Df(0)\,x$ and $F(x)$ are called the linear and nonlinear parts of the system at the origin, respectively. Moreover, we call the linear system*

$$\dot{x} = Df(0)\,x, \tag{5.14}$$

the linearized system at the origin.

Definition 5.12. *Consider system (5.11) with its equilibrium state at the origin. We say that the origin is a hyperbolic equilibrium if it is a hyperbolic equilibrium for the linearized system (see Definition 4.51), i.e. all the eigenvalues of $Df(0)$ have non-zero real parts. An equilibrium which is not hyperbolic is called nonhyperbolic.*

$$\dot{x} = f(x) \qquad \dot{x} = Df(0)\,x$$

Remark 5.13. *We can think of the linearized system as an approximation of system (5.11) near the origin. A natural question that arises here is that to what extent the stability of the linearized system gives information about the stability of the original (nonlinear) system.*

x^*

$\dot{x} = f(x)$

$\dot{y} = g(y)$ - $\qquad g(0) = 0$

$f(x^*)$

$y = x - x^*$

$$\dot{y} = \dot{x} = f(x) = \underbrace{f(y + x^*)}_{g(y)}$$

THEOREM 5.14 (stable and unstable invariant manifolds theorem). *Consider a system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$-smooth. Assume that the origin O of $\mathbb{R}^n$ is a hyperbolic equilibrium. Let E^s and E^u be the stable and unstable subspaces of the linearized system at O, respectively, and suppose $k := \dim(E^s)$. Consider an arbitrary point $x_0 \in \mathbb{R}^n$ and let $x(t)$, where $x(0) = x_0$, be its orbit. Then, there exist manifolds $W^s(O)$ and $W^u(O)$, both containing the origin, such that*

(i) $x(t) \to O$ as $t \to \infty$ if and only if $x_0 \in W^s(O)$.

(ii) $x(t) \to O$ as $t \to -\infty$ if and only if $x_0 \in W^u(O)$.

(iii) $W^s(O)$ and $W^u(O)$ are invariant with respect to the flow of the system and they are tangent to E^s and E^u at O, respectively.

Proof. See [Per01]. □

$W^s(0)$

$W^u(0)$

$h: \mathbb{R}^n \to \mathbb{R}^m$

$\underbrace{\{(x, h(x)) \in \mathbb{R}^{n+m}\}}$

$\dot{x} = f(x)$

0 is hyperbolic

$Df(0)$ <

$Re(\lambda) < 0 \rightsquigarrow E^s$

$Re(\lambda) > 0 \rightsquigarrow E^u$

$W^c(0)$

$\mathbb{R}^n = E^s \oplus E^c \oplus E^u$

$\mathbb{R}^n = E^s \oplus E^u$

$\dim(E^s) = k \leq n$

$W^{cs} \cap W^{cu}$

Definition 5.15. *Consider a $\mathcal{C}^1$ system $\dot{x} = f(x)$ on $\mathbb{R}^n$ with an equilibrium at the origin O. We say O is a sink, source or saddle if it is a sink, source or saddle for the linearized system $\dot{x} = Df(0)x$, respectively.*

- *Recall that O is a sink (resp. source) of $\dot{x} = Df(O)x$ if all the eigenvalues of $Df(O)$ have negative (resp. positive) real parts. It is a saddle if it is hyperbolic, and $Df(O)$ has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part.*

Remark 5.16. *By definition, sinks, sources and saddles are hyperbolic.*

Remark 5.17. *It follows from the stable and unstable invariant manifolds theorem that sinks are asymptotically stable, sources and saddles are unstable.*

$x = Ax$

$\dot{x} = f(x)$

$\dot{x} = Df(0)x$

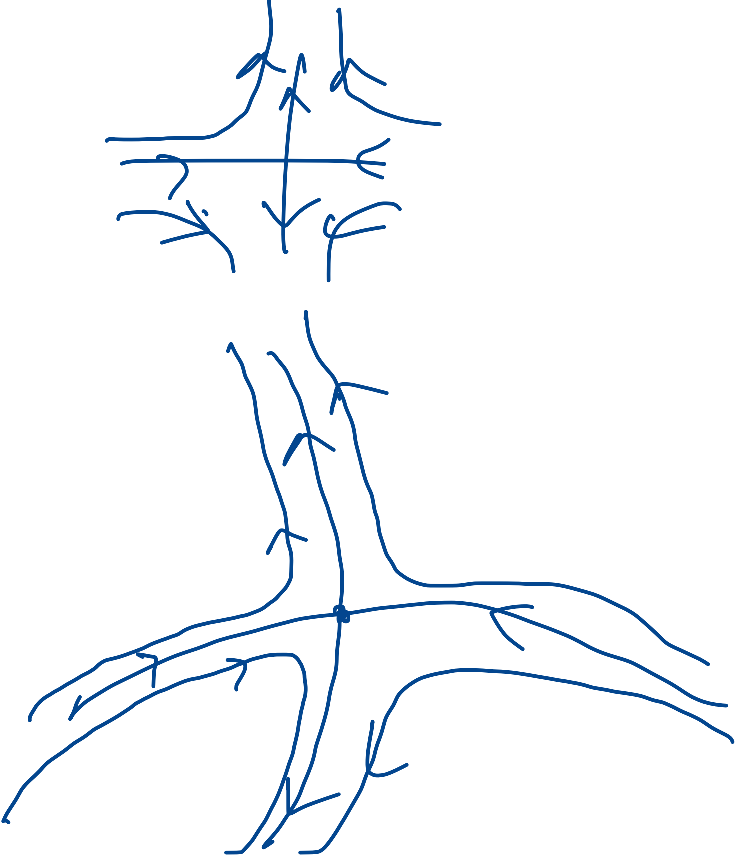
Example 5.18. *Consider the nonlinear system*

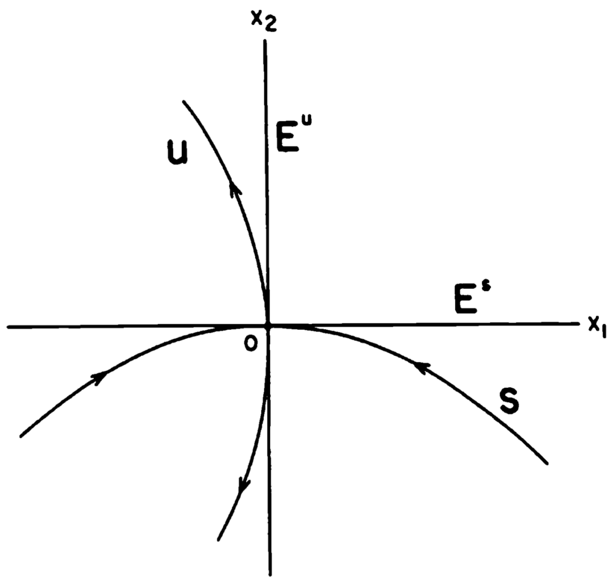
$$\begin{cases} \dot{x}_1 = -x_1 - x_2^2, \\ \dot{x}_2 = x_2 + x_1^2. \end{cases} \tag{5.15}$$

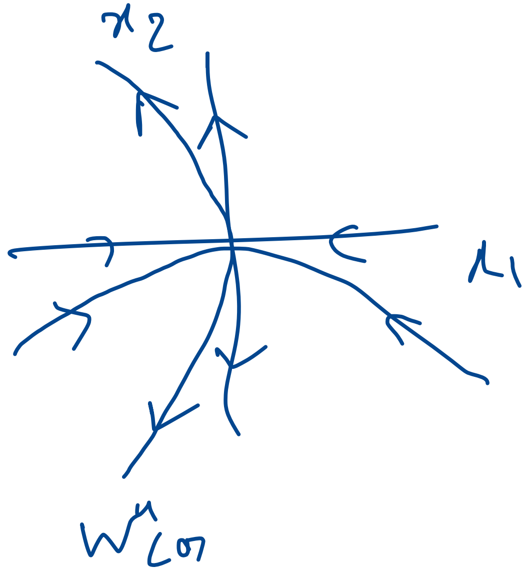
The origin is an equilibrium of this system. The linearized system at the origin is given by

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = x_2. \end{cases} \tag{5.16}$$

The corresponding stable subspace E^s and unstable subspace E^u are the horizontal and vertical axes, respectively. Figure shows the stable and unstable manifolds of the origin of system (5.15) (see [Per01], Example 2 on page 111 for further details). The stable and unstable manifolds are one-dimensional and tangent to E^s and E^u, respectively.







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