$\boxed{\dot{x} = Ax}$ $\quad \begin{pmatrix} \vdots \end{pmatrix}$

$x(0) = x_0$

$e^{At} x_0$

$A \quad \underbrace{2 \times 2}$

4.5. **Stability of equilibria in linear systems**

4.5.1 Preliminaries from Linear Algebra

In this section, we briefly review some concepts from Linear Algebra. For a more detailed review on this topic, we recommed [HSD12] (Sections 2.3 and 5.1).

Vector norms

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a vector in $\mathbb{R}^n$. In this course, we define the norm of x, denoted by $\|x\|$, by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{4.42}$$

This norm is called the standard norm of the Euclidean space $\mathbb{R}^n$.

Remark 4.19. *Norm is a function which assigns a non-negative real number to every vector of* $\mathbb{R}^n$.

Example 4.20. *(i) Consider* $v = (-3, 0, 3, 2) \in \mathbb{R}^4$. *Then*

$$\|v\| = \sqrt{(-3)^2 + 0^2 + 3^2 + 2^2} = \sqrt{9+0+9+4} = \sqrt{22}. \tag{4.43}$$

(ii) Let O *be the origin of* $\mathbb{R}^n$. *Then,* $\|O\| = 0$.

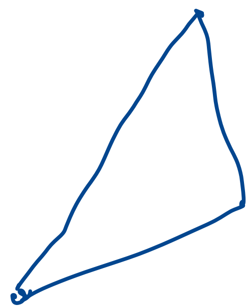
(iii) let $-1 = (-1) \in \mathbb{R}$. *Then* $\|(-1)\| = 1$.

Exercise 4.21. *Prove that the norm defined by (4.42) satisfies the following properties.*

(i) Let O *be the origin of* $\mathbb{R}^n$, *and* $x \in \mathbb{R}^n$ *be an arbitrary vector. Then,* $\|x\| = 0$ *if and only if* $x = O$.

(ii) Let r *be an arbitrary real number, and* v *be an arbitrary vector in* $\mathbb{R}^n$. *Then,* $\|rv\| = |r|\|v\|$.

(iii) (Triangular inequality) Let $x, y \in \mathbb{R}^n$. *Then,* $\|x+y\| \leq \|x\| + \|y\|$



Linear independence

Definition 4.22. *Consider m vectors $v_1, v_2, \dots, v_m$ in $\mathbb{R}^n$. A linear combination of these m vectors is any vector of the form*

$$r_1v_1 + r_2v_2 + \cdots + r_mv_m, \tag{4.44}$$

where r_i are arbitrary real numbers.

Definition 4.23. *Consider the vectors $v_1, v_2, \dots, v_m$, where $m \geq 2$, in $\mathbb{R}^n$. We say that these m vectors are linearly independent if and only if none of these vectors can be written as a linear combination of the other $m-1$ vectors. An equivalent version of this definition is as follows: if $r_1v_1+r_2v_2+\cdots+r_mv_m = 0$, for some real r_i, then $r_1 = \cdots = r_m = 0$.*

Example 4.24. *The vectors $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)$ are linearly independent. Here is why: let $r_1, r_2 \in \mathbb{R}$. Then,*

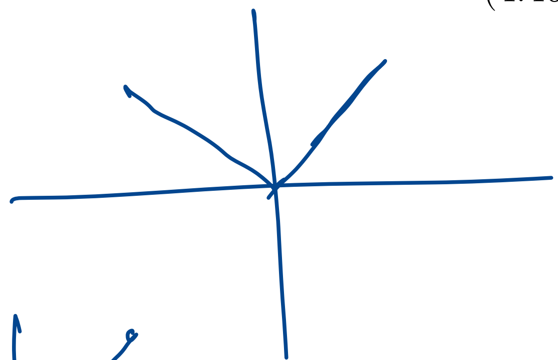
$$r_1\begin{pmatrix}-1\\1\end{pmatrix} + r_2\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Longrightarrow \begin{pmatrix}2r_2 - r_1\\r_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Longrightarrow \boxed{r_1 = 0 \Longrightarrow r_2 = 0.} \tag{4.45}$$

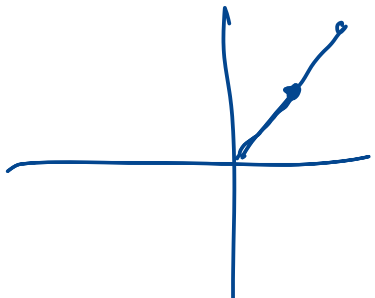
Example 4.25. *The vectors $\left(\begin{smallmatrix}3\\-1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}5\\2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-7\\-5\end{smallmatrix}\right)$ are not linearly independent. Here is why: let $r_1 = -2$, $r_2 = 4$ and $r_3 = 2$. Then,*

$$-2\begin{pmatrix}3\\-1\end{pmatrix} + 4\begin{pmatrix}5\\2\end{pmatrix} + 2\begin{pmatrix}-7\\-5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{4.46}$$

v_1, v_2, v_3

$v_3 = r_1 v_1 + r_2 v_2$





Generated vector subspaces

Suppose that a family of vectors $\{v_\alpha\}$ in $\mathbb{R}^n$ is given. Then, the set

$$V = \{r_1 v_1 + r_2 v_2 + \cdots + r_m v_m : m \geq 1 \text{ is an arbitrary integer, } r_i \text{ are arbitrary real numbers, and } v_i \in \{v_\alpha\}\} \quad (4.47)$$

is a vector subspace of $\mathbb{R}^n$.

Definition 4.26. *The set V is called the vector (sub)space generated by $\{v_\alpha\}$. We denote it by $\langle\{v_\alpha\}\rangle$.*

$\mathbb{R}^n$

$v_1, v_2, \ldots, v_m \in \mathbb{R}^n$

$0 \in V$

$v_i \in V$

Let V_1, V_2, ..., and V_k be subspaces of $\mathbb{R}^n$. Assume that the intersection of any two of these subspaces is only the origin of $\mathbb{R}^n$, i.e. $V_i \cap V_j = \{0\}$, for all $1 \le i, j \le k$. We write

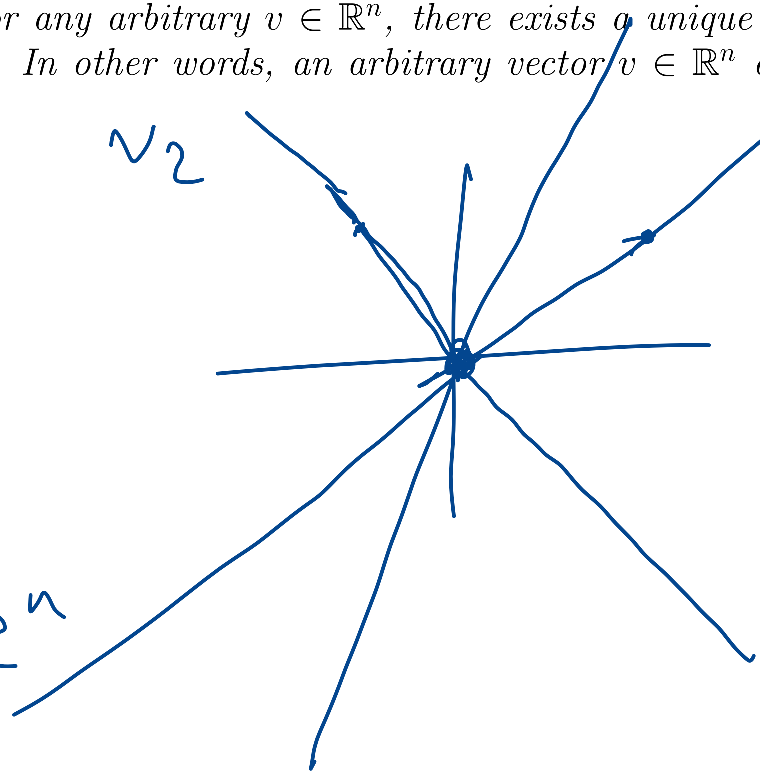
$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k \tag{4.48}$$

$$\sum \dim(V_i) = n$$

if $\mathbb{R}^n$ can be generated by V_1, V_2, ..., V_k, i.e. $\mathbb{R}^n = \langle V_1, \ldots, V_k \rangle$.

Remark 4.27. Assume $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$. Then, for any arbitrary $v \in \mathbb{R}^n$, there exists a unique vector $v_i \in V_i$, for every $i = 1, \ldots, k$, such that $v = v_1 + v_2 + \cdots + v_k$. In other words, an arbitrary vector $v \in \mathbb{R}^n$ can be uniquely decomposed to components in each of the subspaces V_i.

$V_1, V_2, \ldots, V_k \subset \mathbb{R}^n$



$\mathbb{R}^2 = V_1 \oplus V_2$

$\{0\} = V_i \cap V_j \quad \forall i, j$

$\langle V_1, V_2, \ldots, V_k \rangle = \mathbb{R}^n$

$x = x_1 + x_2 + \cdots + x_k$

$\forall x \in \mathbb{R}^n, \ \exists! \, x_i \in V_i$

Eigenvalues, eigenvectors and (generalized) eigenspaces

Definition 4.28. For a given $n \times n$ matrix A, define $p(x) := \det(A - xI)$. The expression $p(x)$ is a polynomial of degree n and is called the characteristic polynomial of A.

Example 4.29. Consider the matrix $Q = \left(\begin{smallmatrix} -1 & 4 \\ 5 & 2 \end{smallmatrix}\right)$. The characteristic polynomial of Q is

$$p(x) = \det(Q - xI) = \det\begin{pmatrix} -1-x & 4 \\ 5 & 2-x \end{pmatrix} = (-1-x) \times (2-x) - 4 \times 5 = x^2 - x - 22. \tag{4.49}$$

Example 4.30. Consider the matrix $Q = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, where a and b are real numbers. The characteristic polynomial of Q is

$$p(x) = \det(Q - xI) = \det\begin{pmatrix} a-x & -b \\ b & a-x \end{pmatrix} = (a-x)^2 + b^2 = x^2 - 2ax + a^2 + b^2. \tag{4.50}$$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$A \longrightarrow p(x)$ n roots

Definition 4.31. *The roots of the characteristic polynomial of the matrix A are called the eigenvalues of A.*

Remark 4.32. *Considering the multiple (repeated) roots of $p(A)$, the $n \times n$ matrix A has exactly n eigenvalues. The number of times that the eigenvalue λ appears as the root of the characteristic polynomial $p(x)$ is called the algebraic multiplicity of λ. An eigenvalue λ is said to be simple if its algebraic multiplicity is 1.*

Example 4.33. *The matrix given in Example 4.29 has two eigenvalues given by $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{89}\right)$.*

Example 4.34. *The matrix given in Example 4.30 has two eigenvalues given by $\lambda_{1,2} = a \pm bi$, where $i = \sqrt{-1}$.*

Remark 4.35. *As Example 4.30 suggests, the eigenvalues of a real matrix A can be non-real too. In this case, non-real eigenvalues appear as pairs. More precisely, if $\lambda = a + bi$ ($i = \sqrt{-1}$) is an eigenvalue of A then the complex conjugate of λ, i.e. $\overline{\lambda} = a - bi$, is an eigenvalue of A too.*

Example 4.36. *It is easily seen that the eigenvalues of diagonal matrices (or more generally, upper or lower triangular matrices) are exactly the elements on the diagonal. For example, each of the matrices*

$$\begin{pmatrix} -7 & & & & & \\ & 5 & & & \mathbf{0} & \\ & & 0 & & & \\ & & & -7 & & \\ & \mathbf{0} & & & 0 & \\ & & & & & -7 \end{pmatrix}, \qquad \begin{pmatrix} -7 & -2 & 21 & 0 & -1 & 0 \\ 0 & 5 & 4 & -4 & 1 & 0 \\ 0 & 0 & 0 & 11 & \sqrt{6} & 8 \\ 0 & 0 & 0 & -7 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3.23 \\ 0 & 0 & 0 & 0 & 0 & -7 \end{pmatrix} \tag{4.51}$$

has six eigenvalues: eigenvalue -7 with algebraic multiplicity 3, eigenvalue 0 with algebraic multiplicity 2, and a simple eigenvalue 5.

Suppose that λ is an eigenvalue of A. Thus, $\det(A - \lambda I) = 0$. Equivalently, $A - \lambda I$ is a singular (noninvertible) matrix. Therefore, $\mathrm{Null}(A - \lambda I)$ is non-trivial (i.e. its dimension is at least 1), where $\mathrm{Null}(A - \lambda I)$ is the null space of $A - \lambda I$. Assume λ is real. Thus, there exists $0 \neq v \in \mathbb{R}^n$ such that $(A - \lambda I)v = 0$. Equivalently, $Av = \lambda v$. Analogously, for the case that λ is non-real, such a vector $0 \neq v \in \mathbb{C}^n$ that satisfies $Av = \lambda v$ can be found.

Definition 4.37. *Let λ be an eigenvalue of A. Any vector $v \neq 0$ that satisfies $Av = \lambda v$ is called an eigenvector of A associated with λ.*

Example 4.38. *Consider the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The characteristic polynomial of A is*

$$p(x) = \det(A - xI) = \det\begin{pmatrix} 2-x & 1 \\ 1 & 2-x \end{pmatrix} = x^2 - 4x + 3. \tag{4.52}$$

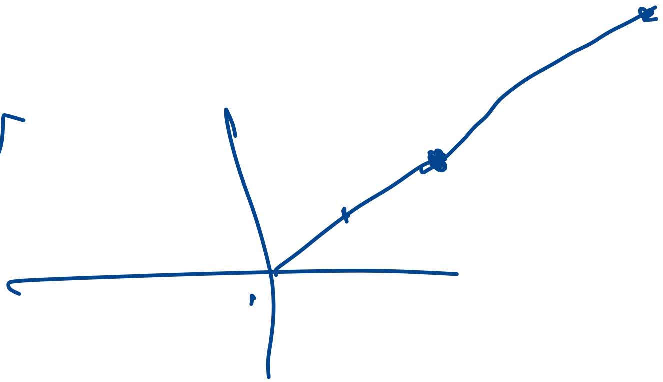
The eigenvalues of A are the roots of $p(x) = x^2 - 4x + 3$ which are $\lambda = 3$ and $\lambda = 1$. For $\lambda = 3$, any vector of the form $v = \left(\begin{smallmatrix} r \\ r \end{smallmatrix}\right)$, where $r \in \mathbb{R}$ is an eigenvector. For instance $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. For $\lambda = 1$, any vector of the form $v = \left(\begin{smallmatrix} r \\ -r \end{smallmatrix}\right)$, where $r \in \mathbb{R}$ is an eigenvector. For instance $v = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$.

$p(\lambda) = \det(A - \lambda I) = 0$

$\underbrace{A - \lambda I}_{\text{singular}} \leadsto \exists v,\ (A - \lambda I)v = 0$

$Av = \lambda I v = \lambda v$

$Av = \lambda v$



$|\lambda| > 1$

$|\lambda| < 1$

PROPOSITION 4.39. *Consider a real $n \times n$ matrix A and suppose that λ is an eigenvalue of it with a corresponding eigenvector v. Then, e^λ is an eigenvalue of e^A and v is an eigenvector of e^A associated with the eigenvalue e^λ.*

Proof. Since v is an eigenvector associated to λ, we have

$$e^A v = \left(I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots \right) v = v + \frac{Av}{1!} + \frac{A^2 v}{2!} + \frac{A^3 v}{3!} + \frac{A^4 v}{4!} + \cdots . \tag{4.53}$$

Notice that, for a positive integer k, we have

$$A^k v = A^{k-1} \underbrace{Av}_{\lambda v} = \lambda A^{k-1} v = \lambda A^{k-2} Av = \lambda^2 A^{k-2} v = \cdots = \lambda^{k-1} Av = \lambda^k v. \tag{4.54}$$

Thus, by (4.53), we have

$$\begin{aligned} e^A v &= v + \frac{Av}{1!} + \frac{A^2 v}{2!} + \frac{A^3 v}{3!} + \frac{A^4 v}{4!} + \cdots \\ &= v + \frac{\lambda v}{1!} + \frac{\lambda^2 v}{2!} + \frac{\lambda^3 v}{3!} + \frac{\lambda^4 v}{4!} + \cdots \\ &= \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots \right) v \\ &= e^\lambda v. \end{aligned} \tag{4.55}$$

□

$Av = \lambda v$

$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + -$

$e^A v$

Generalized eigenvectors and eigenspaces

Having $Av = \lambda v$ is equivalent to $(A - \lambda I)\, v = 0$. The set of all such vectors v is called the eigenspace associated with the eigenvalue λ. We can generalize this concept as follows:

Definition 4.40. *Let λ be an eigenvalue of A. A vector v is said to be a generalized eigenvector if there exists an integer $k \geq 1$ such that $(A - \lambda I)^k\, v = 0$. The set of all such vectors is a vector subspace of $\mathbb{R}^n$ and called the generalized eigenspace associated with the eigenvalue λ.*

Example 4.41. *Consider the matrix $A = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, where $\lambda \in \mathbb{R}$. This matrix has the eigenvalue λ with multiplicity 2. To find the associated eigenvectors $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$, we have*

$$(A - \lambda I)\, v = 0 \Longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow v_2 = 0. \tag{4.56}$$

This suggests that $v = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is an eigenvector associated with λ. Moreover, we cannot find any other independent eigenvector for λ. Let us know try to find generalized eigenvectors. To this end, we need to solve the equation $(A - \lambda I)^2\, v = 0$ for $v \in \mathbb{R}^2$. We have

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow (A - \lambda I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.57}$$

Since $(A - \lambda I)^2 = 0$, any arbitrary vector v satisfies $(A - \lambda I)^2\, v = 0$. Thus, any arbitrary $v \neq 0$ can be considered as a generalized eigenvector. If we look for vectors independent of from the eigenvector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ founded earlier, we can consider, for instance $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

$(A - \lambda I)\, v = 0$ generalized

$m(\lambda) = 10$ $(A - \lambda I)^2 v = 0 \rightarrow$ eigenvector

$(A - \lambda I)^3 v = 0$

4.5.2 Examples of invariant sets for linear flows

LEMMA 4.42. *Let A be a real $n \times n$ matrix and $V \subseteq \mathbb{R}^n$ be a subspace such that $AV \subseteq V$. Then, $e^A V \subseteq V$.*

Proof. Let $x \in V$. Then

$e^A x \in V$

$$e^A x = x + \frac{Ax}{1!} + \frac{A^2x}{2!} + \frac{A^3x}{3!} + \frac{A^4x}{4!} + \cdots \tag{4.58}$$

Define $M_k := x + \frac{Ax}{1!} + \frac{A^2x}{2!} + \cdots + \frac{A^kx}{k!}$. Thus, $e^A x = \lim_{k\to\infty} M_k$. However, since for each positive integer j, we have $A^j x \in V$, we have $M_k \in V$ for all k. However, since $e^A x$ is well-defined, i.e. $\lim_{k\to\infty} M_k$ exists, and V is closed[6], we have that $e^A x \in V$. Thus, $e^A V \subseteq V$. □

$\{M_k\} \to e^A x \in V$

The following proposition is an easy consequence of Lemma 4.42.

$AV \subseteq V$ $\quad x_0$

PROPOSITION 4.43. *Consider a system*

$$\dot{x} = Ax, \tag{4.59}$$

where A is an $n \times n$ real matrix. Let $V \subseteq \mathbb{R}^n$ be a subspace such that $AV \subseteq V$. If x_0 be an arbitrary point in V, we have $e^{tA}x_0 \subseteq V$ for all $t \in \mathbb{R}$. In other words, V is invariant with respect to the flow of system (4.59).

Example 4.44. Consider system (4.59) and suppose λ is an eigenvalue of A. Let E_λ be the generalized eigenspace associated with λ. It can be shown that $AE_\lambda \subset E_\lambda$ (see [Per01], Section 1.9). Then, it follows from Proposition 4.43 that E_λ is invariant with respect to the flow of (4.59), i.e. $e^{At}E_\lambda \subseteq E_\lambda$ for all $t \in \mathbb{R}$.

λ

$E_\lambda = \langle\{v_1, v_2, \ldots, v_k\}\rangle$

[6]Here, closedness is a topological property. A set X in $\mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$, or simply closed, if for any sequence $\{x_k\}$ such that $x_k \in X$ for all k, and $\{x_k\}$ is convergent to some point $x^* \in \mathbb{R}^n$, we have $x^* \in X$. For example, the interval $(0,1]$ is not closed in $\mathbb{R}$ because the sequence $\{\frac{1}{k}\}$ is in $(0,1]$, however, this sequence is convergent to $0 \in \mathbb{R}$ but $0 \notin (0,1]$.

4.5.3 Stable, unstable and center subspaces

Consider a system

$$\boxed{\dot{x} = Ax,} \tag{4.60}$$

where A is an $n \times n$ real matrix. The matrix A has n eigenvalues. Consider the real parts[7] of these eigenvalues. Some of these real parts are positive, some are negative and some equal to zero. Consider all the generalized eigenvectors of these eigenvalues[8].

Definition 4.45. *We define*

(i) $E^s := \Big\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) < 0\} \Big\rangle.$

(ii) $E^c := \Big\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) = 0\} \Big\rangle.$

(iii) $E^u := \Big\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) > 0\} \Big\rangle.$

We call E^s, E^c *and* E^u, *the stable, center and unstable subspaces of system (4.60), respectively.*

$\underline{a} + \underline{b}i$

$$\begin{cases} \underline{\operatorname{Re}(\lambda) < 0} \rightsquigarrow \langle E_{\lambda_1}, \ldots, E_{\lambda_k} \rangle = E^s \\ \operatorname{Re}(\lambda) = 0 \qquad = E^c \\ \operatorname{Re}(\lambda) > 0 \qquad = E^u \end{cases}$$

---

[7]Let $\lambda = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, be a complex number. By the real part and imaginary part of λ, we mean the real numbers a and b, respectively. We write $\operatorname{Re}(\lambda) = a$ and $\operatorname{Im}(\lambda) = b$. For example, $\operatorname{Re}(3-5i) = 3$, $\operatorname{Im}(3-5i) = -5$, $\operatorname{Re}(4i) = 0$, $\operatorname{Im}(4i) = 4$, $\operatorname{Re}(10) = 10$ and $\operatorname{Im}(10) = 0$

[8]If the eigenvalue λ is non-real, its generalized eigenvector can be written as $v = u + iw$, where $u, w \in \mathbb{R}^n$ and $i = \sqrt{-1}$. In such scenario, instead of $v = u + iw \in \mathbb{C}^n$, we consider the vectors u and w individually, and the vector subspace spanned by these two vectors.

Example 4.46. [9] *Consider system (4.60), where A is given by*

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{4.61}$$

The matrix A has the eigenvalues $\lambda_{1,2} = -2 \pm i$ and $\lambda_3 = 3$. This matrix has the eigenvectors

$\lambda_1 = -2+i$

$\lambda_2 = -2-i$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.62}$$

corresponding to $\lambda_{1,2}$, and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.63}$$

corresponding to λ_3. Then

$$E^s = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \tag{4.64}$$

$(x_1 - x_2)$ plane

and

$$E^u = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.65}$$

x_3 - axis

Thus, the stable subspace E^s is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis (see Figure 24).

[9] This example together with its figure is taken from [Per01]

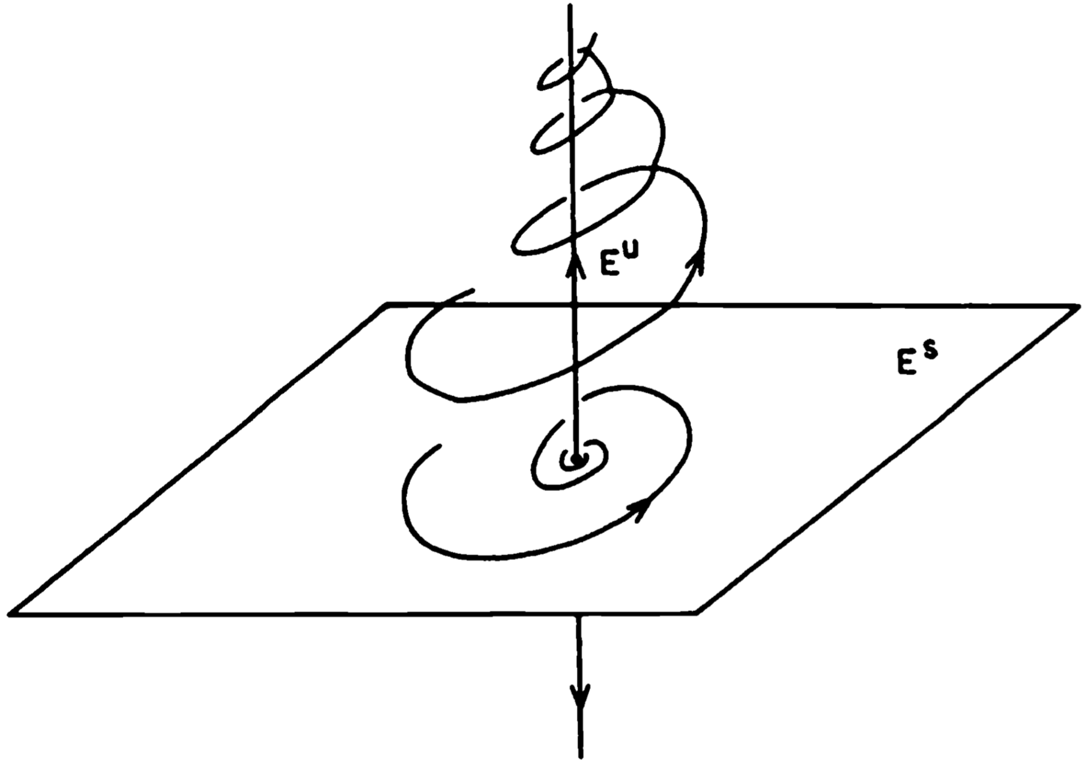


Figure 24: The stable subspace E^s is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis.

Example 4.47. [10] *Consider system (4.60), where A is given by*

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{4.66}$$

The matrix A has the pure imaginary eigenvalue $\lambda_{1,2} = i$ (with multiplicity 2) and $\lambda_3 = 2$. This matrix has the eigenvectors

E^c

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.67}$$

E^u

corresponding to $\lambda_{1,2}$, and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.68}$$

corresponding to λ_3. Then

$$E^{\overset{c}{\cancel{s}}} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \tag{4.69}$$

and

$$E^u = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.70}$$

Thus, the center subspace E^c is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis (See Figure 25).

[10] This example together with its figure is taken from [Per01]

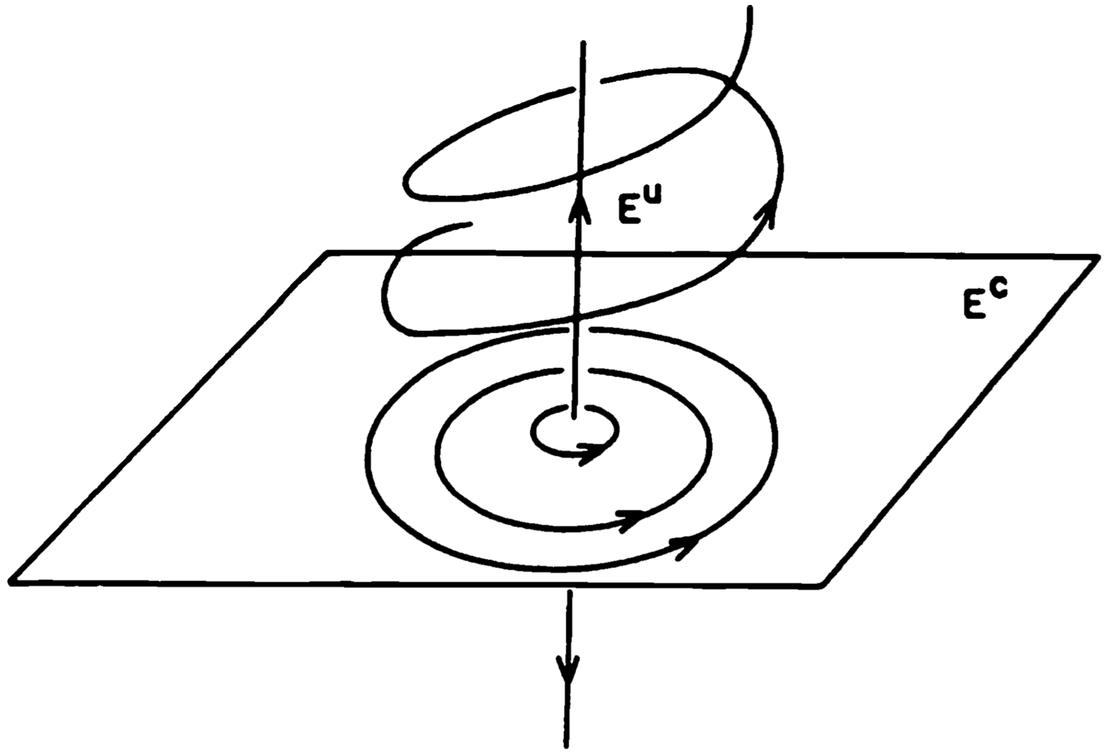


Figure 25: The center subspace E^c is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis.

Example 4.48. *Consider system (4.60), where*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{4.71}$$

and $\lambda \in \mathbb{R}$. In Example 4.41, we discussed that the vectors

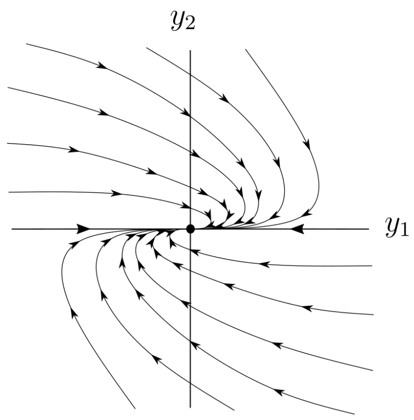
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.72}$$

are the generalized eigenvectors associated with the eigenvalue λ. On the other hand, we have

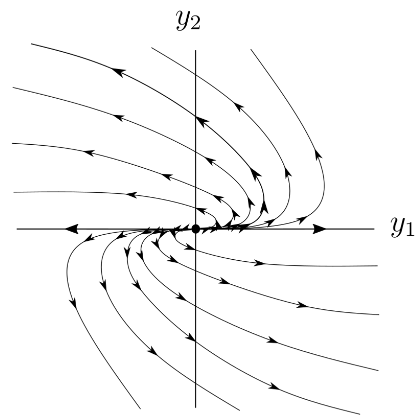
$$\mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.73}$$

Therefore,

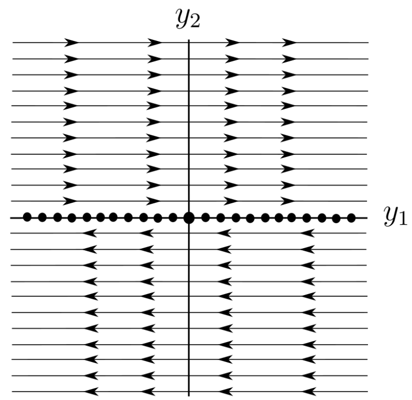
1. *if $\lambda < 0$, then $E^s = \mathbb{R}^2$, $E^c = \{0\}$ and $E^u = \{0\}$,*
2. *if $\lambda = 0$, then $E^s = \{0\}$, $E^c = \mathbb{R}^2$ and $E^u = \{0\}$,*
3. *if $\lambda > 0$, then $E^s = \{0\}$, $E^c = \{0\}$ and $E^u = \mathbb{R}^2$.*



(a) Case $\lambda < 0$: $E^s = \mathbb{R}^2$, $E^c = \{0\}$ and $E^u = \{0\}$.



(b) Case $\lambda > 0$: $E^s = \{0\}$, $E^c = \{0\}$ and $E^u = \mathbb{R}^2$.



(c) Case $\lambda = 0$: $E^s = \{0\}$, $E^c = \mathbb{R}^2$ and $E^u = \{0\}$

Figure 26: Stable, unstable and center spaces of the system $\dot{x} = Ax$, where A is given by (4.71).

THEOREM 4.49. *Consider a system*

$$\boxed{\dot{x} = Ax,} \tag{4.74}$$

where A is a real $n \times n$ matrix. Let E^s, E^u and E^c be the stable, unstable and center subspaces of the system. Then each of these three spaces are invariant with respect to the flow of system (4.74). Moreover, we have

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c. \tag{4.75}$$

$v = v^s + v^u + v^c$ $\quad e^{At} v = e^{At} v^s + e^{At} v^u + e^{At} v^c$

Proof. See [Per01]. □

PROPOSITION 4.50. [11] *Let O be the origin of $\mathbb{R}^n$. Consider a point $x_0 \in \mathbb{R}^n$, and let E^s, E^u and E^c be the stable, unstable and center subspaces of system (4.74), respectively. Then, the following hold.*

(i) If $x_0 \in E^s$, then $e^{tA}x_0 \to O$ as $t \to \infty$.

(i) If $x_0 \in E^u$, then $e^{tA}x_0 \to O$ as $t \to -\infty$.

(i) If $e^{tA}x_0 \to O$ as $t \to \infty$, then $x_0 \in E^s \oplus E^c$. E^u

E^s $\quad x_1 - a -$

(i) If $e^{tA}x_0 \to O$ as $t \to -\infty$, then $x_0 \in E^u \oplus E^c$.

Proof. See [Per01]. □

$O(e^{-|t|a})$

[11] A more comprehensive version of this result is valid: E^s (resp. E^u) is indeed the set of all points x_0 in $\mathbb{R}^n$ that converge to O exponentially fast as $t \to \infty$ (resp. $t \to -\infty$), i.e. $\exists\, a > 0$, $M > 0$ such that $\|e^{tA}x_0\| \leq Me^{-a|t|}$ for $t \geq 0$ (resp. $t \leq 0$). On the other hand, E^c is the set of the points $x_0 \in \mathbb{R}^n$ whose orbits grow at most sub-exponentially fast as $t \to \pm\infty$, i.e. $\forall a > 0$, $\frac{\|e^{tA}x_0\|}{e^{a|t|}} \to 0$ as $t \to \pm\infty$ (see [Rob98], Theorem 6.1).

Definition 4.51. *Let O be the origin of $\mathbb{R}^n$ and consider system (4.74).*

(i) We say the equilibrium O is hyperbolic if A has no eigenvalue with zero real part, i.e. $E^c = \{O\}$, or equivalently $\mathbb{R}^n = E^s \oplus E^u$. Otherwise, we say O is nonhyperbolic, i.e. A has at least one eigenvalue with zero real part.

(ii) We say the equilibrium O is a sink (resp. source) if all the eigenvalues of A have negative (resp. positive) real parts, i.e. $E^s = \mathbb{R}^n$ (resp. $E^u = \mathbb{R}^n$).

(iii) We say the equilibrium O is a saddle if it is hyperbolic, and the matrix A has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part, i.e. $\mathbb{R}^n = E^s \oplus E^u$, $\dim(E^s) \geq 1$ and $\dim(E^u) \geq 1$.

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