Lecture Notes On

Synchronization From A Mathematical Point Of View

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Contents

1. Introduction to the course

1. **Practical information:**

   - All the lectures will be held on **Tuesdays, 15-17 (GMT+3)**.
   - All the lectures will be recorded.
   - All the necessary information and updates on the course (including the lecture notes and recorded videos) will be posted on the virtual learning system of Kadir Has university (Hub).

2. **Materials for further reading:**

   The present lecture notes together with the recorded videos of the lectures is sufficient for this course. However, if you are interested to study dynamical systems further and in more details, there are so many books available that you can use. Below are what we recommend.

   - If you're not into reading a full textbook and prefer something short, we recommend
     - S. Van Strien, *Lecture notes on ODEs.* available for free on the author's webpage.
   - If you want to read a textbook and have some background in mathematics (e.g. mathematical analysis, linear algebra and calculus), we recommend
     - L. Perko, *Differential equations and dynamical systems*, third edition.
     - M. W. Hirsch, S. Smale and R. L. Devaney, *Differential equations, dynamical systems and an introduction to chaos*, third edition.
   - If you want to read a textbook but you don't feel comfortable reading math literature or you prefer a textbook with more taste of applications, we recommend
     - S. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*, second edition.

- In the introductory session, we saw examples of synchronization in real world phenomena.
- A mathematical model for these phenomena is given by

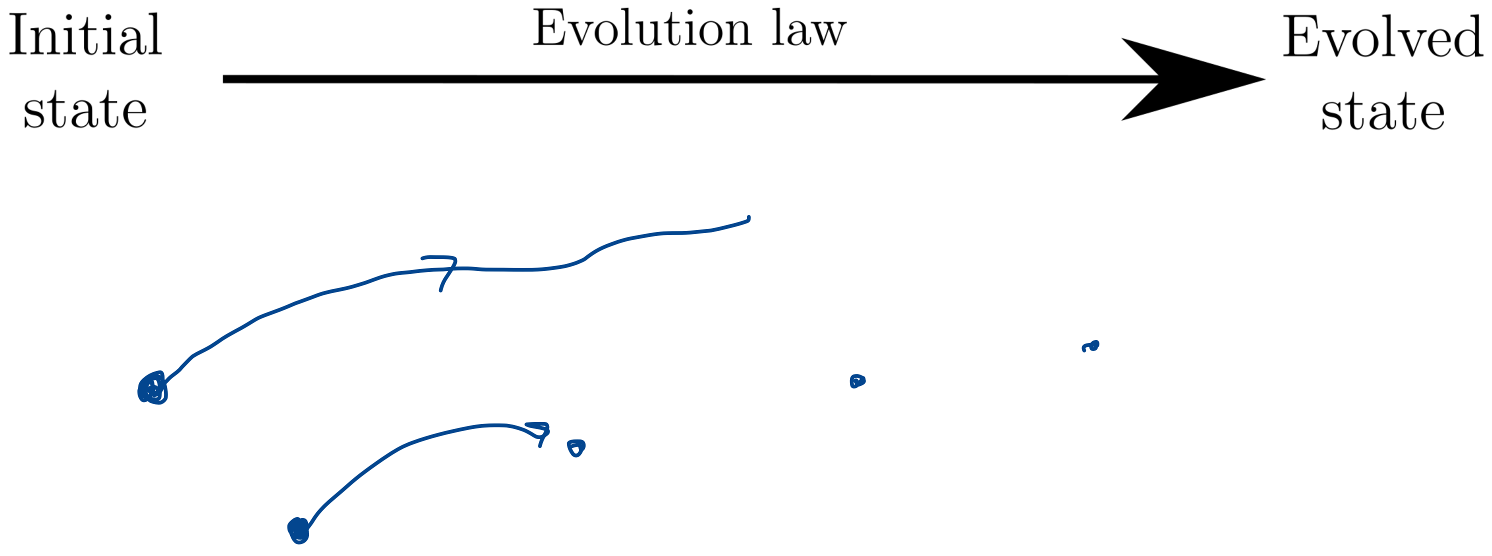
$$\frac{dx_i}{dt} = f_i\left(x_i\right) + \alpha \sum_{j=1}^{N} A_{ij} H_i\left(x_j - x_i\right), \qquad \forall i \in \{1, \dots, N\}, \tag{1.1}$$

  where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2\left(\mathbb{R}^n\right)$.
- Our main goal in this course is to develop methods that help us to understand the dynamics of this mathematical model.

2. Introduction to dynamical systems

2.1. Definition

- Dynamical systems studies the evolution of a system.
  - A dynamical system is defined by a law of evolution which involves **time** and **state (position)**. For a given initial state, this evolution law describes how this state evolves as time passes.
  - This rule can be deterministic or random.
  - Time can vary continuously or discretely.
- In this course, we focus on deterministic[1] continuous(-time) systems.



[1] A system is called deterministic if the entire past and future of a state are uniquely determined by its state at the present time. [Arn92]

- Rigorous formulation:

Definition 2.1. *Consider* $\mathbb{R}^n$ *($n \geq 1$). Let t be real and x be a point in $\mathbb{R}^n$. A dynamical system is a function*

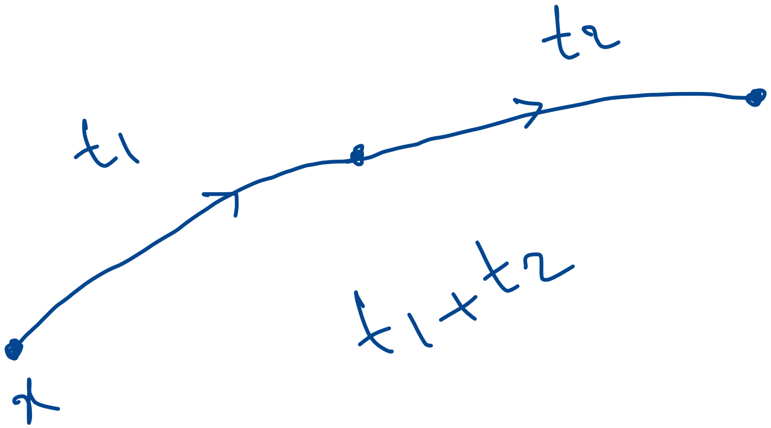
$$\left\{\begin{aligned} \phi :&\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \\ &(t, x) \mapsto \phi(t, x) \end{aligned}\right. \tag{2.1}$$

that satisfies

$$\left\{\begin{aligned} &\textit{(i)}\ \phi(0, x) = x \text{ for all } x \in \mathbb{R}^n. \\ &\textit{(ii)}\ \phi\left(t_2, \underline{\phi\left(t_1, x\right)}\right) = \phi\left(\underline{t_1 + t_2}, x\right) \text{ for all } x \in \mathbb{R}^n \text{ and for arbitrary } t_1, t_2 \in \mathbb{R}. \end{aligned}\right.$$

These two conditions are known as flow properties.

- The variable t is called the time variable. The variable x is called the phase or state variable. We also call $\mathbb{R}^n$ the phase space.

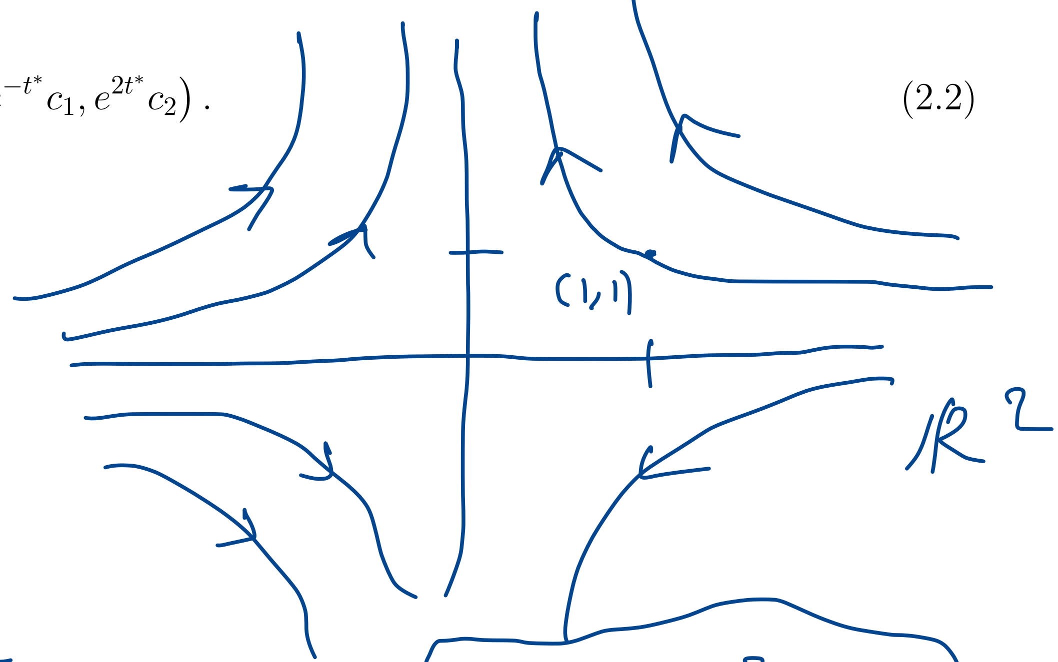


2.2. Visualization of dynamical systems

Suppose a dynamical system $\phi(t, x)$ is given. A standard way to visualize this dynamical system is that for all $x \in \mathbb{R}^n$, we draw the trajectory curve (path) that x takes as t varies. We show the direction of increasing in time by an arrow on this curve.

Example 2.2. One can show (see Exercise 2.4) that the function $\boxed{\phi(t, x) = \left(e^{-t}x_1, e^{2t}x_2\right),}$ *where* $x = (x_1, x_2) \in \mathbb{R}^2$ *is the phase variable, is a dynamical system. Consider an arbitrary point* $(c_1, c_2) \in \mathbb{R}^2$*. Suppose* (x_1, x_2) *is a point on the trajectory of* (c_1, c_2)*. Thus, there exists* $t^* \in \mathbb{R}$ *such that*

$$(x_1, x_2) = \left(e^{-t^*}c_1, e^{2t^*}c_2\right). \tag{2.2}$$



$(t, (x_1, x_2)) \longmapsto (e^{-t}x_1, e^{2t}x_2)$

$t=0 \curvearrowright (c_1, c_2) \in \mathbb{R}^2$

$t=t^* \curvearrowright (x_1, x_2)$

$$\left.\begin{aligned} x_1 &= e^{-t^*}c_1 \\ x_2 &= e^{2t^*}c_2 \end{aligned}\right\} \Rightarrow \left(\frac{c_1}{x_1}\right)^2 = e^{2t^*} = \frac{x_2}{c_2} \Rightarrow \boxed{x_2 = c_2c_1^2 \cdot \frac{1}{x_1^2}}$$

$$x_2 = \frac{1}{x_1^2}$$

In Example 2.2, if $c_1, c_2 \neq 0$, we have $\frac{x_1}{c_1} = e^{-t^*}$ and $\frac{x_2}{c_2} = e^{2t^*}$. Thus $\frac{x_2}{c_2} = \left(\frac{c_1}{x_1}\right)^2$. This implies that the curve paths through (c_1, c_2), where $c_1, c_2 \neq 0$, is given by

$$x_1^2 x_2 = c_1^2 c_2. \tag{2.3}$$

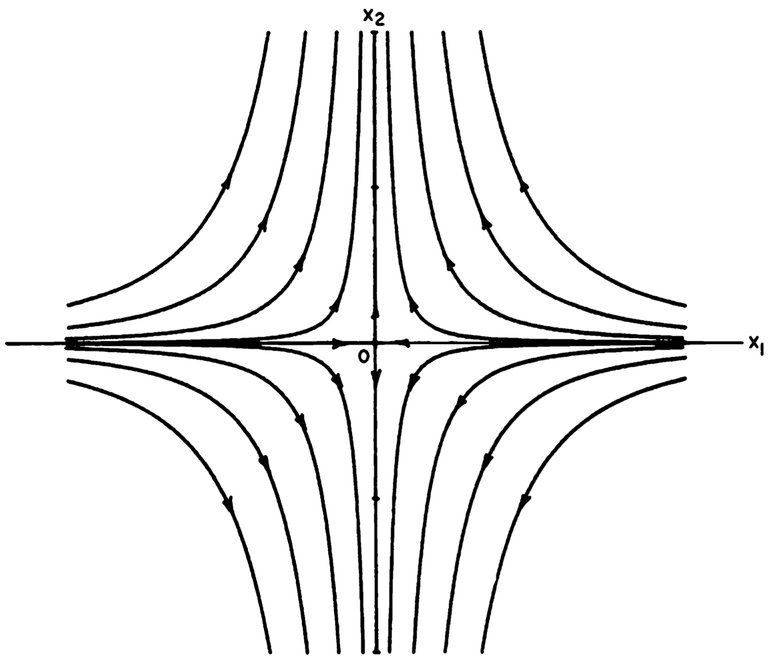


Figure 1: This figure shows how points in $\mathbb{R}^2$ move by $\phi(t,x) = \left(e^{-t}x_1, e^{2t}x_2\right)$

2.3. An example of dynamical systems

$x \in \mathbb{R}$

Example 2.3. Let a be a real number. Consider the function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by $\phi(t,x) = e^{at}x$. We show that ϕ satisfies the flow properties.

(i) $\phi(0,x) = e^{a\times 0}x = x$.

(ii) For arbitrary real t_1 and t_2, we have $\phi\left(t_2, \phi\left(t_1,x\right)\right) = \phi\left(t_2, e^{at_1}x\right) = e^{at_2} \times e^{at_1}x = e^{a(t_1+t_2)}x = \phi\left(t_1+t_2, x\right)$.

$\{0\}$

$\{x>0\}$

$\{x<0\}$

(a) Case $a < 0$.

(b) Case $a > 0$.

Figure 2: Visualization of the dynamical system $\phi(t,x) = e^{at}x$

Exercise 2.4. Determine whether or not the following functions ϕ satisfy the flow properties.

1. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t,x) = t + x$.
2. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t,x) = t^2 + x$.
3. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t,x) = tx$.
4. $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, *defined by*
$$\phi\left(t,x\right) = \left(e^{at}x_1, e^{bt}x_2\right), \tag{2.4}$$
where a and b are real constants, and $x = (x_1, x_2) \in \mathbb{R}^2$ is the phase variable.

2.4. **Orbits**

- Fix a point x_0 in the phase space $\mathbb{R}^n$. The path that x_0 takes as time t varies is called the orbit or trajectory of x_0. More precisely, the orbit or trajectory of x_0 is the set

$$\{\phi(t, x_0) : \quad t \in \mathbb{R}\}. \tag{2.5}$$

$\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

x_0

  – Geometrically, an orbit is a curve in the phase space.

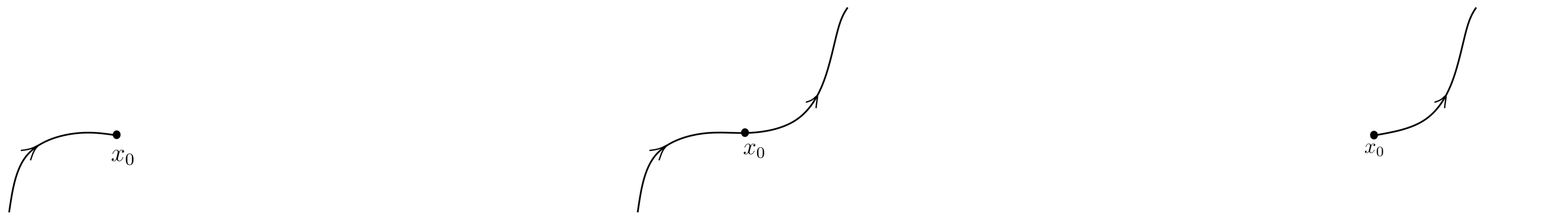
- The orbit of x_0 is defined for both positive and negative times t. However, for a given orbit, we can also focus only on positive or negative times:

  – The forward orbit or positive semi-orbit of a point $x_0 \in \mathbb{R}^n$ is the set

$$\{\phi(t, x_0) : \quad t \geq 0\}. \tag{2.6}$$

  – The backward orbit or negative semi-orbit of a point $x_0 \in \mathbb{R}^n$ is the set

$$\{\phi(t, x_0) : \quad t \leq 0\}. \tag{2.7}$$



(a) The backward orbit of x_0. (b) The orbit of x_0. (c) The forward orbit of x_0.

Example 2.5. *For the dynamical system given in Example 2.3, there are three orbits: (i)* $\{x : x > 0\}$ *(ii)* $\{0\}$ *(iii)* $\{x : x < 0\}$.

Exercise 2.6. *Consider the dynamical system given in Example 2.3 and let* $a = 0$. *How many orbits does this dynamical system have?*

Remark 2.7. *Orbits of a dynamical system never cross (here is why: assume the contrary. Thus, two different orbits Γ_1 and Γ_2, where $\Gamma_1 \neq \Gamma_2$, have a common point p. Then, $\Gamma_1 = \{\phi(t, p) : t \in \mathbb{R}\} = \Gamma_2$, which is a contradiction).*

- Some important examples of orbits:

  (i) Equilibria:

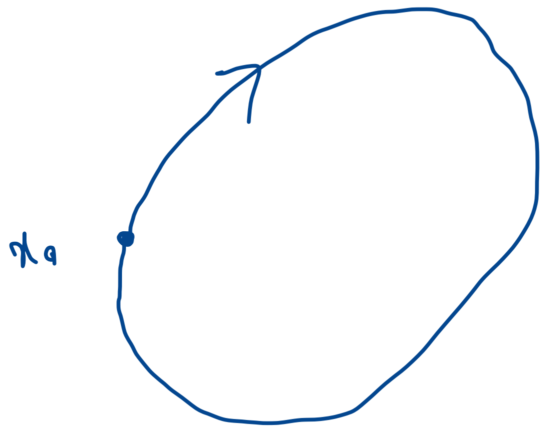
  - The orbit of a point x_0 is said to be constant if it contains only the point x_0 itself, i.e. the entire orbit is just the single point $\{x_0\}$.
  - We have $\phi(t, x_0) = x_0$ for all $t \in \mathbb{R}$. In other words, the point x_0 is steady; it does not move!
  - When the orbit of x_0 is constant, we call the point x_0 an equilibrium point or steady state (also called fixed point in some literatures).

  $\{x_0\}$

  (ii) Periodic orbits

  - The orbit $\phi(t, x_0)$ of x_0 is said to be periodic if there exists $T > 0$ such that $\phi(t, x_0) = \phi(t + T, x_0)$.
  - The point x_0 comes back to itself after passing time T.

- We call the set of all the orbits of a dynamical system the *phase portrait* of that dynamical system. However, loosely speaking, by phase portrait we usually mean the visualization of that phase portrait, i.e. drawing figures like Figures 1 and 2.



$T > 0$

$\varphi(0, x_0) = x_0$

$\varphi(T, x_0) = x_0$

2.5. **Time-t maps**

- Consider again the function

$$\begin{aligned} \phi :&\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \\ &(t, x) \mapsto \phi(t, x). \end{aligned} \tag{2.8}$$

- We can think of two particular scenarios here:
  1. We fix x and allow t to vary.
  2. We fix t and allow x to vary.
- Scenario 1:
  - This is the scenario that we considered before.
  - Let $x = x_0 \in \mathbb{R}^n$. In this case,

$$\begin{aligned} \phi :&\mathbb{R} \to \mathbb{R}^n, \\ &t \mapsto \phi(t, x_0). \end{aligned} \tag{2.9}$$

  - The function ϕ maps a real variable t to a point in $\mathbb{R}^n$. In particular, it maps 0 to x_0.
  - $\phi(t, x_0)$, as t varies in $\mathbb{R}$, describes the orbit of the point x_0.

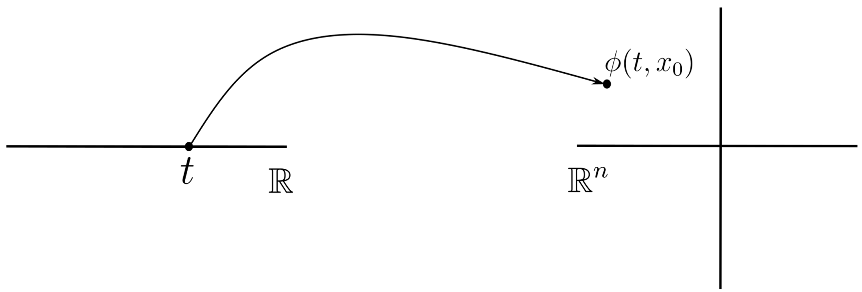


Figure 4: For a fixed $x = x_0$, the function ϕ maps $\mathbb{R}$ to $\mathbb{R}^n$.

- Scenario 2:
  - Let $t = t_0 \in \mathbb{R}$. In this case,

$$\begin{aligned} \phi :&\mathbb{R}^n \to \mathbb{R}^n, \\ &x \mapsto \phi(t_0, x). \end{aligned} \tag{2.10}$$

  - The function ϕ maps a point in $\mathbb{R}^n$ to a point in $\mathbb{R}^n$. In particular, when $t_0 = 0$, the function ϕ maps each point to itself, i.e. ϕ is the identity map.
  - When the time variable t is fixed, the function ϕ is called *time-t map*. For example, $x \mapsto \phi(1, x)$ is called *time*-1 *map*.
  - Time-t maps become important when we want to discretize a continuous-time system.

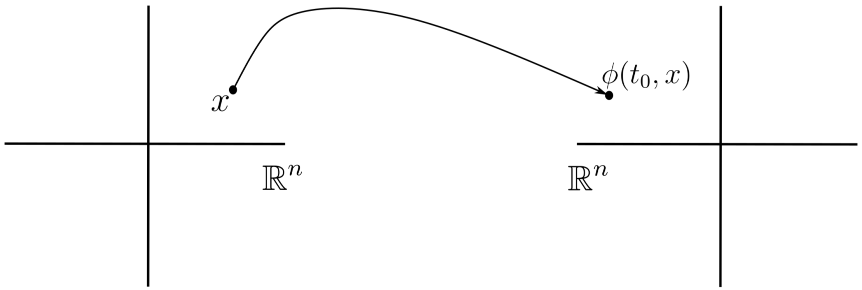


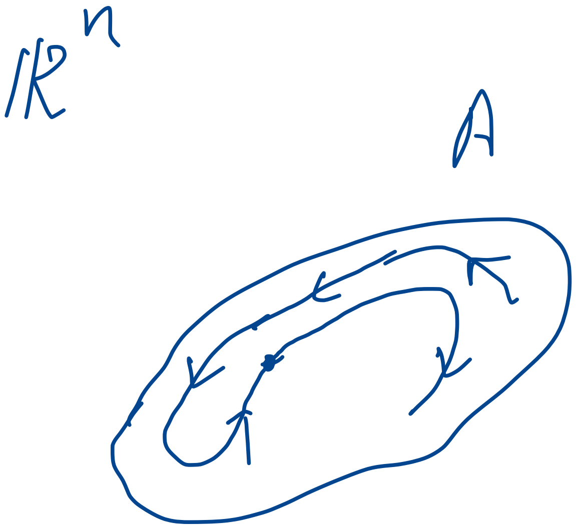
Figure 5: For a fixed $t = t_0$, the function ϕ maps $\mathbb{R}^n$ to $\mathbb{R}^n$.

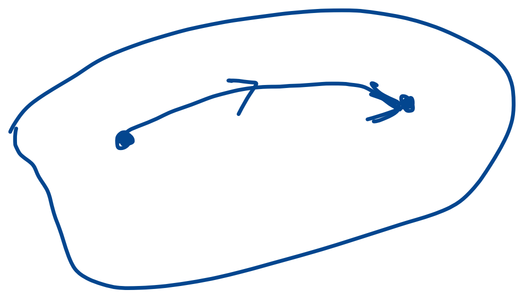
2.6. **Invariance**

Consider a dynamical system $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and let $A \neq \emptyset$ be a subset of $\mathbb{R}^n$.

- We say A is invariant with respect to ϕ if for every point x_0 in A, the entire orbit of x_0 lies in A, i.e. $\phi(t, x_0) \in A$ for all $t \in \mathbb{R}$.
  - The set A is invariant if and only if when we start from a point in A, moving forward and backward both, we remain in A and never leave it.

Example 2.8. Let x_0 be an arbitrary point of the phase space. The orbit of x_0 is an invariant set.

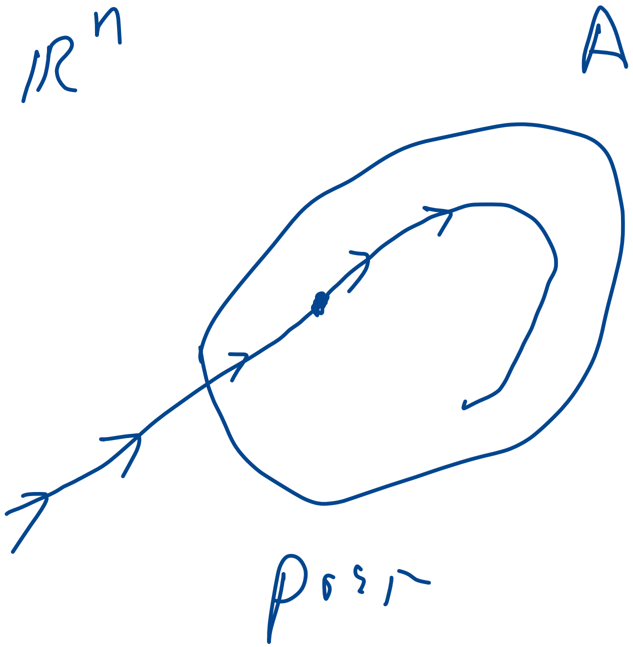


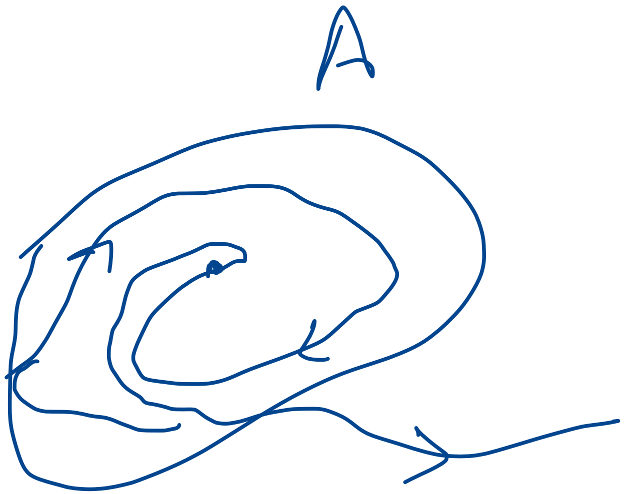


- We say A is positively invariant or forward invariant with respect to ϕ if for every point x_0 in A, the forward orbit of x_0 lies entirely in A, i.e. $\phi(t, x_0) \in A$ for all $t \geq 0$.
  - The set A is invariant if and only if when we start from a point in A and move forward, we remain in A and never leave it.
- We say A is negatively invariant or backward invariant with respect to ϕ if for every point x_0 in A, the backward orbit of x_0 lies entirely in A, i.e. $\phi(t, x_0) \in A$ for all $t \leq 0$.
  - The set A is invariant if and only if when we start from a point in A and move backward, we remain in A and never leave it.

Remark 2.9. *Every invariant set is forward and backward invariant as well. However, not every forward or backward invariant set is necessarily invariant.*

Exercise 2.10. *Determine whether or not the sets* $A_1 = (-2, 1)$ *and* $A_2 = (2, 3)$ *in* $\mathbb{R}$ *are (positively or negatively) invariant with respect to the dynamical system given by Example 2.3.*





3. Introduction to ODEs

3.1. Vector fields

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$. One can think of f as

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}, \tag{3.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$.

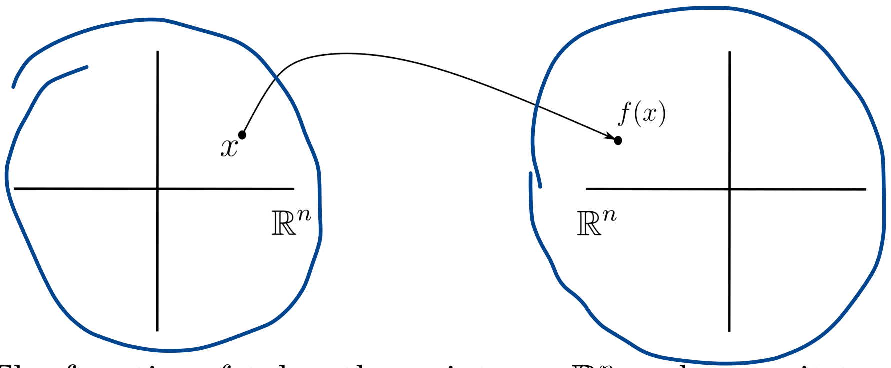


Figure 6: The function f takes the point $x \in \mathbb{R}^n$ and maps it to $f(x) \in \mathbb{R}^n$.

Example 3.1. *The followings are examples of $f : \mathbb{R}^n \to \mathbb{R}^n$.*

(i) $f(x) = x^2 + 1$. *Here,* $f : \mathbb{R}^1 \to \mathbb{R}^1$. *(ii)* $f\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 + \sin x_2 \\ x_2 \end{smallmatrix}\right)$. *Here,* $f : \mathbb{R}^2 \to \mathbb{R}^2$.
(iii) $f\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \begin{pmatrix} g(x_1) + \alpha H(x_2 - x_1) \\ g(x_2) + \alpha H(x_1 - x_2) \end{pmatrix}$, *where α is a real constant, and $g, H : \mathbb{R}^2 \to \mathbb{R}$. Here, $f : \mathbb{R}^2 \to \mathbb{R}^2$.*
(iv) $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. *Here,* $f : \mathbb{R}^3 \to \mathbb{R}^3$.
(v) $f(x) = \alpha g(x) + \beta h(x)$, *where $x \in \mathbb{R}^n$, α and β are real constants and $g, h : \mathbb{R}^n \to \mathbb{R}^n$. Here, $f : \mathbb{R}^n \to \mathbb{R}^n$.*

- In this course, we call a function $f : \mathbb{R}^n \to \mathbb{R}^n$ a vector field. Here is why:
- One way to visualize a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is that for every point $x \in \mathbb{R}^n$, we draw the vector $f(x)$ starting at the point x and ending at $x + f(x)$ (see Figure 7).

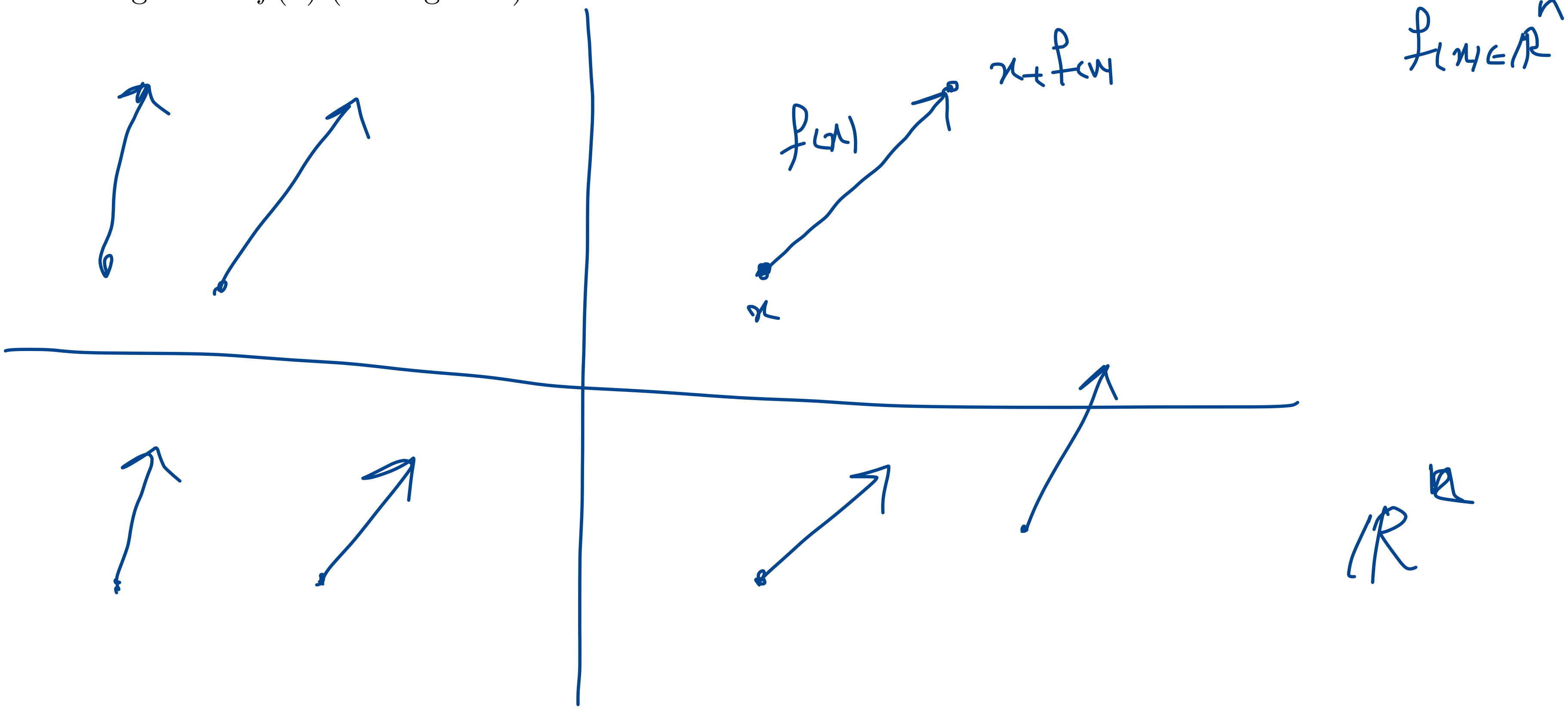


Figure 7: For every point $x \in \mathbb{R}^n$, we draw the vector $f(x)$ starting at the point x and ending at $x + f(x)$.

$\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

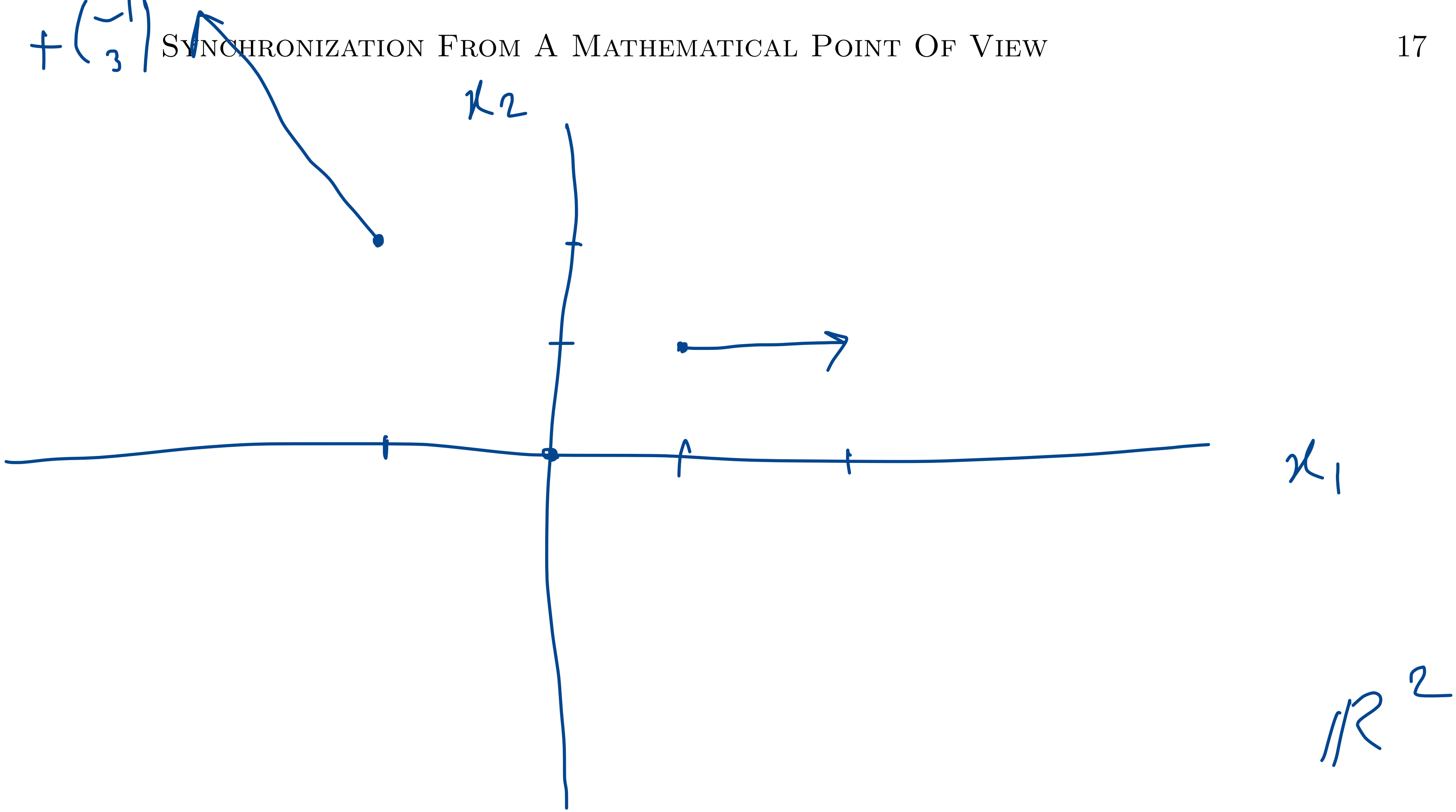


Figure 8: The vector field $f(x_1, x_2) = (x_1, x_2 - x_1)$.

$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $\quad f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

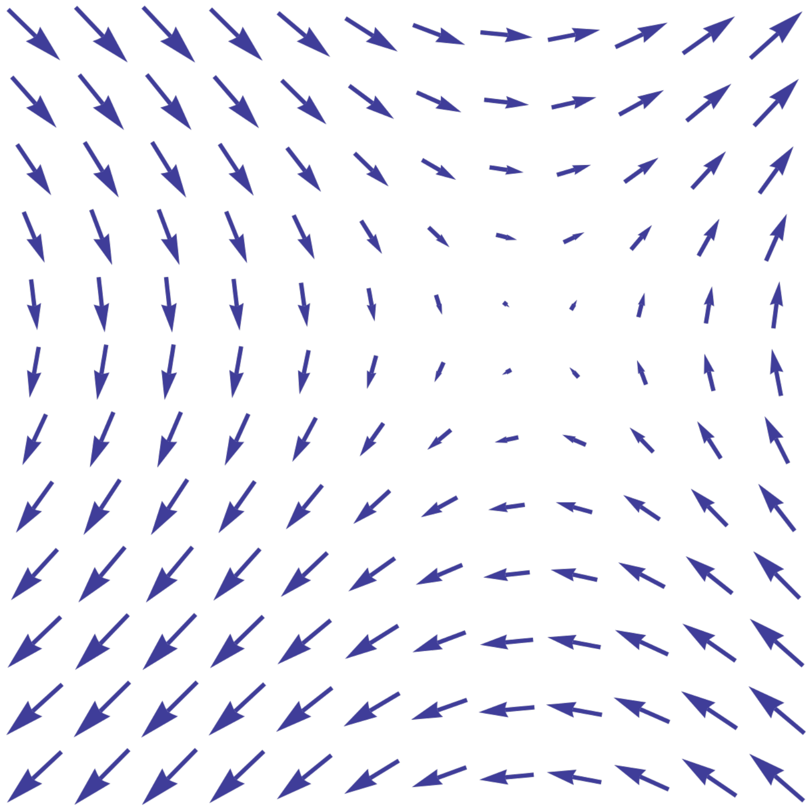


Figure 9: A portion of the vector field $f(x_1, x_2) = (\sin x_2, \sin x_1)$ on $\mathbb{R}^2$ (this figure is copied from Wikipedia).

3.2. **Solutions of ODEs**

- **Question:** Suppose that $f:\mathbb{R}^n \to \mathbb{R}^n$ is given. Let $x_0 \in \mathbb{R}^n$. Does there exist any function

$x \quad t \mapsto x(t) \in \mathbb{R}^n$

$$\begin{aligned} x : & \mathbb{R} \to \mathbb{R}^n \\ & t \mapsto x(t) = (x_1(t), \ldots, x_n(t)) \end{aligned} \tag{3.2}$$

such that $\frac{dx(t)}{dt} = f(x(t))$ and $x(0) = x_0$? If it exists, is it unique?

- By $\frac{dx(t)}{dt} = f(x(t))$, we mean

$$\begin{cases} \dfrac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ \dfrac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad \vdots \quad\ \vdots \qquad\qquad \vdots \\ \dfrac{dx_n(t)}{dt} = f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \tag{3.3}$$

where $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$.

- Some terminologies and notations:
  - We call the equation $\frac{dx(t)}{dt} = f(x(t))$, i.e. equation (3.3), a system of ordinary differential equations.
  - The condition $x(0) = x_0$ is called an initial condition.
  - The equation $\frac{dx(t)}{dt} = f(x(t))$ together with the initial condition $x(0) = x_0$ is called an initial value problem (I.V.P).
  - Such a function $x(t)$, if it exists, is called a solution of the initial value problem $\frac{dx(t)}{dt} = f(x(t))$ and $x(0) = x_0$.
  - In this course, for simplicity, we use dot to show derivative with respect to time. For example, $\dot{x} := \frac{dx(t)}{dt}$.

$f : \mathbb{R} \to \mathbb{R}$

Example 3.2. The initial value problem

$$\dot{x} = 3x^{\frac{2}{3}}, \qquad \text{and} \quad x(0) = 0 \qquad f(x) = 3x^{\frac{2}{3}} \tag{3.4}$$

has two different solutions $x(t) = t^3$ *and* $x(t) = 0$.

$x(t) = t^3 \Rightarrow \dot{x} = 3t^2 = 3(t^3)^{\frac{2}{3}} = 3x^{\frac{2}{3}}$

$x(0) = 0$

Example 3.3. The initial value problem

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = -4x_2, \\ \dot{x}_2 = x_1, \qquad \text{and} \quad x(0) = (c_1, c_2), \end{cases} \tag{3.5}$$

where (c_1, c_2) *is an arbitrary point in* $\mathbb{R}^2$ *has at least one solution (we will see later that this is the only solution) defined for* $t \in \mathbb{R}$, *given by*

$$\begin{cases} x_1(t) = c_1 \cos 2t - 2c_2 \sin 2t, \\ x_2(t) = \dfrac{c_1}{2} \sin 2t + c_2 \cos 2t. \end{cases} \tag{3.6}$$

Example 3.4. The initial value problem

$(-\infty \ 1)$ $\cancel{(1 \ \infty)}$

$$\dot{x} = x^2, \qquad \text{and} \quad x(0) = 1 \tag{3.7}$$

has the solutions $x(t) = \frac{1}{1-t}$, *which is defined for* $t \in (-\infty, 1)$. *Notice that* $x(t) = \frac{1}{1-t}$ *satisfies* $\dot{x} = x^2$ *for* $t \in (1, \infty)$, *however the initial condition is not satisfied since* $0 \notin (1, \infty)$.

Example 3.5. The initial value problem

$$\dot{x} = f(x), \qquad \text{and} \quad x(0) = 0, \tag{3.8}$$

where

$$f(x) = \begin{cases} 1 & \text{when } x < 0 \\ -1 & \text{when } x \geq 0 \end{cases} \tag{3.9}$$

has no solutions. Can you see why? Hint: if $x(t)$ *is a solution then it needs to be differentiable at every* t, *particularly at* $t = 0$.

- Remember the question that we asked earlier: Does the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$ have solution? Uniqueness?
  - **Quick Answer:** As the examples that we just reviewed suggest:

    In general, NO! For an arbitrary vector field f and arbitrary initial point $x_0 \in \mathbb{R}^n$, the solutions neither need to exist nor be unique; even if they exist, they are not necessarily defined for all $t \in \mathbb{R}$.

- Before we proceed to an elegant answer to our question, let's see what the geometrical/physical meaning of a solution is. Suppose that there is a unique solution $x(t)$ for the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$.
  - The solution $x(t)$ describes how x_0 moves in $\mathbb{R}^n$ as t varies.
  - Define $\Gamma := \{x(t) : t \in \mathbb{R}\}$. Geometrically, Γ is a curve in $\mathbb{R}^n$. Let $t^* \in \mathbb{R}$ and $x^* := x(t^*)$. The tangent vector to the curve Γ is given by $\frac{dx}{dt}(t^*)$. However, $\frac{dx}{dt}(t^*) = f(x(t^*)) = f(x^*)$. This means that at every point x on the curve Γ, the vector $f(x)$ is tangent to Γ.
  - Having in mind that $x(t)$ describes the movement of x_0, the vector $f(x^*)$ is the velocity vector at time t^*.

$x(t)$

$$\left.\frac{d\,x(t)}{dt}\right|_{t=t^*}$$

$$\dot{x} = f(x)$$

$$= f(x(t^*)) = f(x^*)$$

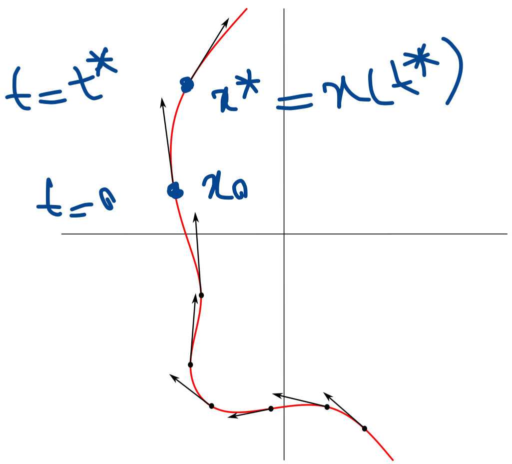


Figure 10: At every point x_0, the solution curve of $\dot{x} = f(x)$ passing through x_0 is tangent to the vector $f(x_0)$.

THEOREM 3.6. *Let $f \in \mathcal{C}^1(\mathbb{R}^n)$[2], and $x_0 \in \mathbb{R}^n$. Then, there exists a open interval $I_{x_0} = (\alpha(x_0), \beta(x_0))$, where $\alpha(x_0) < 0 < \beta(x_0)$, such that the initial value problem*

$\dot{x} = f(t, x) \quad x = (x_1, -, x_n)$

$\dot{x}_{n+1} = 1$

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{3.10}$$

has a unique solution $x(t)$ on I_{x_0}. Moreover, the interval I_{x_0} is maximal in the sense that if $x^*(t)$ is a solution of (3.10) defined on an interval J, then $J \subset I_{x_0}$ and $x^*(t) = x(t)$ on J.

Proof. See [VS18], the proof of the existence-uniqueness theorem (Theorem 3.6) and the discussion on the maximal solutions (Chapter 5). □

Remark 3.7. *This theorem guarantees that if the vector field is $\mathcal{C}^1$-smooth, then the solution of the I.V.P exists and is defined on some maximal interval $I \subseteq \mathbb{R}$. However, as Example 3.4 shows, this interval is not necessarily equal to $\mathbb{R}$; although this theorem guarantees the existence and uniqueness of the solution, it does not guarantee the solution to exist for all $t \in \mathbb{R}$. In this course, we assume*[3] *that the solution $x(t)$ of the I.V.P (3.10) exists for all $t \in \mathbb{R}$, i.e. $I_{x_0} = \mathbb{R}$.*

Remark 3.8. *In system (3.10), the function f does not depend directly on t. Such systems are called autonomous. Nonautonomous systems are those where t is an independent variable of the function f; a nonautonomous system is written as $\dot{x} = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Theorem 3.6 holds for nonautonomous case too (see [VS18], Theorem 3.6). In this course, our focus is on autonomous systems.*

Exercise 3.9. *Can you say why Theorem 3.6 cannot guarantee the existence and uniqueness of solutions in Examples 3.2 and 3.5? What can this theorem say about Example 3.3?*

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[2]In general, we say $f : \mathbb{R}^n \to \mathbb{R}^m$, given by $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}$, is $\mathcal{C}^1$-smooth, denoted by $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$, if for all $1 \le i \le m$ and $1 \le j \le n$, the partial derivative $\frac{\partial f_i}{\partial x_j}(x_1, \cdots, x_n)$ exists and is continuous. When $n = m$, we write $f \in \mathcal{C}^1(\mathbb{R}^n)$. The functions f in Example 3.1 are smooth (assuming g, h and H are smooth).

[3]This assumption is not that much strong. Indeed, for any arbitrary system $\dot{x} = f(x)$, there exists a system of ODEs which is topologically equivalent to $\dot{x} = f(x)$ and its solutions are defined on whole $\mathbb{R}$ (see [Per01], Section 3.1).

In general, finding explicit solutions of ODEs is not possible. Even when the explicit solutions are available, they can be very difficult to deal with. The aim of this course is not solving ODEs. In this course, we develop methods that can be used to analyze ODEs without necessarily solving them.

3.3. **Dynamical systems defined by ODEs**

Theorem 3.6 states that when $f \in \mathcal{C}^1(\mathbb{R}^n)$, for any arbitrary $x_0 \in \mathbb{R}^n$, the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$ has a unique solution on I_{x_0}. Denote this solution by $\phi_t(x_0)$.

Solving this I.V.P for every $x_0 \in \mathbb{R}^n$, we obtain a family of solutions $\phi_t(x_0)$. Define

$$\phi(t,x) := \phi_t(x). \tag{3.11}$$

Then

THEOREM 3.10. *The function $\phi : (t,x) \mapsto \phi(t,x)$ defined by (3.11) satisfies the flow properties*

(i) $\phi(0,x) = x$ *for all* $x \in \mathbb{R}^n$.

(ii) $\phi(t_2, \phi(t_1, x)) = \phi(t_1 + t_2, x)$ *for all* $x \in \mathbb{R}^n$ *and for arbitrary* $t_1, t_2 \in \mathbb{R}$.

Remark 3.11. *Assuming $I_{x_0} = \mathbb{R}$ for every $x_0 \in \mathbb{R}^n$, Theorem 3.10 implies that the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by (3.11) is a dynamical system (see Definition 2.1).*

References

[Arn92] V. Arnold. *Ordinary Differential Equations.* Springer Verlag Textbook, third edition, 1992.

[Per01] L Perko. *Differential equations and dynamical systems.* Springer-Verlag, third edition, 2001.

[VS18] S Van Strien. *Lecture notes on ODEs.* available at https://www.ma.imperial.ac.uk/ svanstri/Files/de-4th.pdf, Spring 2018.