Advanced Calculus

Limit, Boundary and Isolated Points, Dense Sets

ThinkBS: Basic Sciences in Engineering Education

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Limit Points

We define a deleted neighborhood [with its centered removed] as $N_r^*(x) = N_r(x) \setminus \{x\}$.
Assume $A \subseteq (X, d)$. A point $x \in X$ is called a limit point (or an accumulation point) of A if for all $r > 0$, $N_r^*(x) \cap A \neq \emptyset$.
The set of limit points of a set A is called the derived set and is shown by A'.
A set is which is both close and every point of it is a limit point, is called **perfect**.

Example: 0 is a limit point of the set $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. (why?) In fact 0 is the only limit point of this set. The set of cluster points is equal to $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. This set is neither an open set, nor a close one.

Example: In a metric space, $\bar{A} = A \cup A'$.

Boundary and Isolated Points

Assume $A \subseteq (X, d)$. A point $x \in X$ is called a boundary point of A if $x \in \bar{A} \cap \bar{A}^c$. The set of boundary points of A is shown by $bd(A)$ or $\partial(A)$.

A point $a \in A$ is called an isolated point if it is not a limit point of A.

Example 1: $\partial((0,1)) = \{0,1\}$.

Example 2: 1 is an isolated point for $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Bounded and Dense Sets, Separable Space

Assume $A \subseteq (X, d)$. A is called bounded if there is $x \in X$ and $r > 0$ such that $A \subseteq N_r(x)$.

A is called dense in X if every point of X is a limit point of A, or a point of A (or both).

A metric space (X, d) is called separable, if it contains a countable dense subset.

Example 1: $(0, 1) \subseteq \mathbb{R}$ is bounded but $\mathbb{Q} \subseteq \mathbb{R}$ is not.

Example 2: $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$. The set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$. (why?)

Example 3: $\mathbb{R}$ with Euclidean metric is separable but with discrete metric is not. (why?)

Unions and Intersections of Open and Close Sets

In a metric space:

- An arbitrary union of open sets, is an open set.
- A finite intersection of open sets, is an open set (provide a counterexample for an infinite one).
- An arbitrary intersection of close sets, is a close set.
- A finite union of close sets, is a close set (provide a counterexample for an infinite one).

For further properties of metric spaces, see 'Chap. 2: Metric Spaces'.