Advanced Calculus

Metric Spaces: Definition

ThinkBS: Basic Sciences in Engineering Education

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We saw before that some properties of functions (like intermediate value property) is not only dependent on the function itself, but also on the underlying setup that function is defined on.

Here we will try to study such properties of spaces: Is there enough points in the space for a sequence (a function from $\mathbb{N}$ to that space) to have some properties like obtaining a limit? How we can formalize 'limit' and convergence? How we can define how far points from each other to talk about 'convergence'?

We will start by defining what a metric space is.

Metric Spaces: Definition

Let X be a nonempty set and assume that $d\colon X \times X \to \mathbb{R}$ satisfies the following conditions for all $x, y, z \in X$:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$ (symmetric property),
4. $d(x, y) \leq d(x, z) + d(y, z)$ (triangle inequality).

Then d is a called a metric on X and (X, d) is called a metric space.

Metric Spaces: examples

Example 1: $\mathbb{R}$ with $d(x,y) = |x-y|$ is a metric space (why?)

Example 2: $\mathbb{R}^n$ with $d_p(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$ is a metric space with $1 \leq p \in \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n)$. For $p = 2$, $\mathbb{R}^n$ is called Euclidean space.
To see this one needs to know the following facts:

- Assume $0 < p, q \in \mathbb{R}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for every positive real number u and v we have $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$ and equality holds iff $u^p = v^q$.
- (**Hölder's inequality**) Assume $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, also $0 < p, q \in \mathbb{R}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then
$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}}$$
For $p = q = 2$ this inequality is called the Cauchy–Bunyakovsky–Schwarz inequality.
- (**Minkowski's inequality**) Assume $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $0 \leq p \in \mathbb{R}$. Then
$$(\sum_{i=1}^{n} |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}$$
.

Metric Spaces: examples

Example 3: Let X be any non-empty set with

$$d(x, y) = \begin{cases} 1 & \textit{if} \quad x \neq y \\ 0 & \textit{otherwise} \end{cases}$$

is a metric space (why?). This metric is called the discrete metric on X.

Example 4: $\mathbb{R}^n$ with $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric space. This metric is called Taxicab or Manhattan metric.

Example 5: $\mathbb{R}^n$ with $d(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ is a metric space.