Advanced Calculus

Sequencess and Convergence

ThinkBS: Basic Sciences in Engineering Education

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Sequencess

A sequence in a set X is a function $a : \mathbb{N} \to X$ (or sometimes from $\mathbb{N} \cup \{0\}$). Instead of writing $a(n)$, we usually use a_n for the n-th element of the sequence, and use $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$ to refer to the sequence itself.

The difference between the set $\{a_n \mid n \in \mathbb{N}\}$ and the sequence $\{a_n\}_{n\in\mathbb{N}}$ is that when we talk about the sequence, we actually talk about the set $\{(n, a_n) \mid n \in \mathbb{N} \; and \; a_n \in X\}$, and thus an innate "this one comes after that one" is assumed.

Assume (X, d) is a metric space. We say that the sequence $\{a_n\}$ converges to a and show it by $a_n \to a$ if for any $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ we have $d(a_n, a) < \varepsilon$.

In this case we write $\lim_{n\to\infty} a_n = a$.

Loosely speaking, a convergent sequence is a sequence for which the elements get closer and closer to a point in the space after some step.

Example 1: $\{a_n = \frac{1}{n}\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges to 0. Assume that an arbitrary $\varepsilon > 0$ is given. According to the archimedean property of $\mathbb{R}$, we can find an $N \in \mathbb{N}$ such that $\varepsilon N > 1$ (why?). Thus for every $n \geq N$, we have:

$$d(\frac{1}{n}, 0) = |\frac{1}{n} - 0| \leq \frac{1}{N} < \varepsilon.$$

Example 2: The sequence $\{a_n = \frac{1}{n}\}_{n\in\mathbb{N}}$, in the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a metric space in itself with metric *inherited* from $\mathbb{R}$, is not convergent. There is no such point as 0 in this space!

For further properties of sequences, see 'Chap. 3: Convergent Sequences'.