Advanced Calculus

Introduction to Calulus on Manifolds

ThinkBS: Basic Sciences in Engineering Education

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Calculus on Other Spaces

Up to now, we have discussed the theories of differentiation and integration in Euclidean spaces $\mathbb{R}^n$. What we want to do is to try to generalize these concepts to spaces different than Euclidean ones. But since we know a lot about Euclidean spaces, we prefer for those spaces to be somehow locally similar to Euclidean one such that we can transfer the tools there and do calculus on such objects as well. These spaces are called "manifolds".

Charts

Let X be a topological space. An n-dimensional chart on X is a homeomorphism (continuous invertible function with continuous inverse) $h : U \to U'$ from an open subset $U \subseteq X$, the chart domain, onto an open subset $U' \subseteq \mathbb{R}^n$.

If every point in X belongs to some chart domain of X, the space X is called locally Euclidean.

It is often useful to include the name of the chart domain in the notation for the chart and speak of the chart as (U, h).

Transition Map

If (U, h) and (V, k) are two n-dimensional charts on X, then the homeomorphism $k\ o\ (h^{-1}|_{h(U \cap V)})$ from $h(U \cap V)$ to $k(U \cap V)$ is called the change of charts map, or transition map, from h to k. If it is not only a homeomorphism but a diffeomorphism (differentiable invertible function with differentiable inverse), we say that the two charts are differentiably related.

By differentiable, in the sense of analysis in $\mathbb{R}^n$, we always mean of class $\mathcal{C}^\infty$: having continuous partial derivatives of all orders.

In particular, a homeomorphism f between open sets in $\mathbb{R}^n$ is a diffeomorphism if and only if both f and f^{-1} are $\mathcal{C}^\infty$ functions.

Atlases and Differentiable Structures

A set of n-dimensional charts on X whose chart domains cover all of X is an n-dimensional atlas on X. The atlas is differentiable if all its charts are differentiably related, and two differentiable atlases $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent if $\mathfrak{A} \cup \mathfrak{B}$ is also differentiable.

Denote by $\mathcal{D}(\mathfrak{A})$ the set of all the charts (U, h) on X that are differentiably related to all the charts in $\mathfrak{A}$. The set of charts $\mathcal{D}(\mathfrak{A})$ is thus an n-dimensional differentiable atlas and a maximal one: every chart we could have added without destroying differentiability is already there.

An n-dimensional differentiable structure on a topological space X is a maximal n-dimensional differentiable atlas.

Hausdorff and Second Countable Spaces

A topological space is called Hausdorff if for any two different points there exist disjoint neighborhoods. Every metric space is Hausdorff (why?)

a topological space (X, τ) is second-countable if there exists some countable collection $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ of open subsets of τ such that any open subset of τ can be written as a union of elements of some subfamily of $\mathcal{U}$. A second-countable space is said to satisfy the second axiom of countability.

Differentiable Manifolds

An n-dimensional differentiable manifold is a pair $(M, \mathcal{D})$ consisting of a Hausdorff and second countable space M and an n-dimensional differentiable structure $\mathcal{D}$ on M.

A function $f : M \to \mathbb{R}$ is differentiable (i.e. $\mathcal{C}^\infty$) at $p \in M$ if for some chart (U, h) around p, the downstairs function $f \ o \ h^{-1}$ is differentiable in a neighborhood of $h(p)$.