Advanced Calculus

Mean Value Theorems

ThinkBS: Basic Sciences in Engineering Education

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Mean Value Theorems

Before coming to Mean Value Theorems (MVT) we first need to give the following definition:

Assume $f : I \to \mathbb{R}$ and $x_0 \in I^\circ$. We say that f has a local maximum (resp., local minimum) at x_0 if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ (resp., $f(x) \geq f(x_0)$) for all $x \in (x_0 - \delta, x_0 + \delta) \cap I$.

We say that f has a local extremum at x_0 if it has a local maximum or a local minimum at x_0.

f is said to have a global or absolute maxima/minima if the conditions specified above is true for all $x \in I$.

Let $f : I \to \mathbb{R}$ and $x_0 \in I^\circ$ be an interior point. If f has a local extremum at x_0 and is differentiable at x_0 then $f'(x_0) = 0$.

Fermat's Theorem states that the tangent line to the graph of f at the point $(x_0, f(x_0))$ is horizontal.

The converse of this theorem is not true. For instance consider $f(x) = x^3$ at point $x_0 = 0$. While $f'(0) = 0$ but $x_0 = 0$ is not a maximum or minimum point.

Darboux and Rolle's Theorem

Darboux's Theorem: Let $f : I \to \mathbb{R}$ be differentiable on I° and let $a < b$ in I° be such that $f'(a) < \eta < f'(b)$. Then there exists $\xi \in (a, b)$ such that $f'(\xi) = \eta$.

Rolle's Theorem: Let Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, be differentiable on (a, b) and satisfy $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Find an example which shows that c in the Rolle's theorem is not necessarily unique.

Mean Value Theorem (MVT)

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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To interpret MVT geometrically, assume that $f(t)$ represents a car's position at time t, then $f'(t)$ represents the instantaneous velocity at that time, and $\frac{f(b)-f(a)}{b-a}$ represents the average velocity over the time interval $[a, b]$. Thus the MVT implies that, at some time $c \in (a, b)$ the instantaneous velocity is in fact equal to the average (i.e., mean) velocity.

Corollaries of MVT

Corollary 1: Let $h > 0$ and suppose that $f : [x, x + h] \to \mathbb{R}$ is continuous on $[x, x + h]$ and differentiable on $(x, x + h)$. Then there exists a number $\theta \in (0, 1)$ such that

$$f(x + h) = f(x) + hf'(x + \theta h)$$

Corollary 2: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If for all $x \in (a, b)$, we have $f'(x) = 0$, then f is a constant function; there is a $c \in \mathbb{R}$ such that $f(x) = c$.

Corollaries of MVT

Corollary 3: Let $f : I \to \mathbb{R}$ be continuous on I and differentiable on its interior I°.

Then f is increasing (resp., strictly increasing) on I if for all $x \in I^\circ$, $f'(x) \geq 0$ (resp., $f'(x) > 0$).

Similarly, f is decreasing (resp., strictly decreasing) on I if for all $x \in I^\circ$, $f'(x) \leq 0$ (resp., $f'(x) < 0$).

L'Hôpital's Rule

Let $-\infty \leq a < b \leq \infty$; and let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$.

Suppose that either

- $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or
- $\lim_{x\to a} g(x) = \infty$

If for some $L \in [-\infty, \infty]$ we have

- $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$

then we also have:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Use the L'Hôpital's rule to calculate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.