Advanced Calculus

Compactness

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Compact Sets

An open cover of a set A in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $A \subseteq \cup_\alpha G_\alpha$.

A subset K of a metric space X is said to be **compact** if every open cover of K contains a finite subcover.

This means that if $\{G_\alpha\}$ i s an **arbitrary** open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

Example: $\mathbb{R}$ is not compact: The open cover $\{(-n, n)\}_{n \in \mathbb{N}}$ has no finite subcover.

Theorems About Compact Sets

- Every compact set in a metric space is close.
- Every compact set in a metric space is bounded.
- Any close subset of a compact set is itself compact.
- Image of a compact set under a continuous function is a compact set.
- (Heine–Borel theorem) In Euclidean spaces $\mathbb{R}^n$, a subset is compact if and only if it is both bounded and close.
- Assume $\{a_n\}$ is convergent to a in a metric space (X, d). Then the set $\{a_n \mid n \in \mathbb{N}\} \cup \{a\}$ is compact in X.
- Any continuous function on a compact set, is uniformly continuous.

Other Formulations of Compactness

Before giving equivalent statements for compactness, we need to define the following concepts:

- A metric space (X, d) is called **totally bounded** iff for every real number $\varepsilon > 0$, there exists a finite collection of open neighborhoods in X of radius ε whose union contains X.
- Assume $\{a_n\}_{n=1,2,3,\ldots}$ is a sequence in X and consider $n_1 < n_2 < n_3 < \ldots$ to be a strictly increasing sequence of natural numbers. Then, $\{a_{n_i}\}_{i=1,2,3,\ldots}$ is called a **subsequence** of $\{a_n\}_{n=1,2,3,\ldots}$.
- We say a family $\{A_\alpha\}_{\alpha \in J}$ has **finite intersection property**, whenever the intersection of a finite arbitrary sub-family is not empty. We say $\{A_\alpha\}_{\alpha \in J}$ has **total intersection property**, whenever the intersection of all of its members is not empty.

Other Formulations of Compactness

- (Sequential Compactness) A metric space (X, d) is compact iff every sequence in X has a convergent subsequence.
- (Bolzano - Weierstrass Compactness) A metric space (X, d) is compact iff every infinite subset of X has a limit point.
- A metric space (X, d) is compact iff it is complete and totally bounded.
- (Cantor's Intersection Theorem) A metric space (X, d) is compact iff any family $\{F_\alpha\}_{\alpha \in J}$ of its close sets with finite intersection property, has total intersection property.

For further information, see 'Chap. 2: Compact Sets'.