min-max, inf-sup

Definition (Minimum and Maximum Elements) Let $(X, \leq)$ be a nonempty ordered set with the order relation $\leq$ and $A \subseteq X$. An element $a \in A$ is called a *minimum* (*maximum*) if for all $x \in A$ we have $a \leq x$ ($x \leq a$).

- What is the minimum and maximum of the set $\mathbb{N}_k = \{1, 2, \ldots, k\}$?
- What is the minimum and maximum of the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$?

Definition (Lower and Upper Bounds) Let $(X, \leq)$ be a nonempty ordered set with the order relation $\leq$ and $A \subseteq X$. An element $b \in X$ is called a *lower bound* (*upper bound*) for A if for all $x \in A$ we have $b \leq x$ ($x \leq b$).

Definition (Infimum and Supremum) The number α is called the *infimum* (*supremum*) of the non-empty set A if it is the greatest lower bound (least upper bound) of A.

Theorem (Characteristic properties of supremum) Prove that the necessary and sufficient conditions for α to be the supremum of a nonempty set A is:

i. for every $x \in A$, $x \leq \alpha$.

ii. for every number u less than α, A has an element greater than u.

Formulate and prove the same theorem for the infimum.

Theorem (ε formulation of the characteristic properties of supremum) Prove that the necessary and sufficient conditions for α to be the supremum of a nonempty set A is:

i. for every $x \in A$, $x \leq \alpha$.

ii. for every positive number ε, A has an element x such that $\alpha - \varepsilon < x$.

Formulate and prove the same theorem for the infimum.

For the sets given below, decide on if the sets have minimum, maximum, lower or upper bound and infimum or supremum.

a. $\{x \in \mathbb{R} \mid 0 < x\}$

b. $\{x \in \mathbb{R} \mid 0 \leq x\}$

c. $\{x \in \mathbb{R} \mid x^2 + x + 1 > 0\}$

d. $\{x \in \mathbb{R}^+ \mid x + x^{-1} > 0\}$

e. $\{\frac{1-5n}{n+1} \mid n \in \mathbb{N}\}$

f. $\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 - \frac{1}{n}\right]$

g. * $\{\frac{m}{n} + \frac{4n}{m} \mid m, n \in \mathbb{N}\}$

Please send your answers to “ayse.bilge@khas.edu.tr”.

References

[1] Hungerford, T. W. (1980). Algebra, volume 73 of. Graduate Texts in Mathematics.