What is a zero-divisor?

For convenience, let us remind you what a group is:

Definition (Group) A non-empty set G together with a binary operation $+ \colon G \times G \to G$, called 'addition' (but it is not necessarily the addition that we know of, it is just a symbol to denote the group operation), is called a *Group*, and is shown by $(G, +)$, if the following properties are satisfied:

1. For every x, y, $z \in G$, $(x + y) + z = x + (y + z)$ (Associativity),
2. There exist an element $0 \in G$ such that for all $x \in G$, $0+x = x+0 = x$ (Identity element),
3. For every $x \in G$ there is a $y \in G$ such that $x + y = y + x = 0$ (Inverse element),

- In addition to above conditions, if for a group G it happens that for all x, $y \in G$, $x + y = y + x$, then the group is called *Abelian.*

What happens if we put another operation on a Group?

Definition The group $(G, +)$ with the second operator called multiplication, denoted by $* \colon G\backslash\{0\}\times G\backslash\{0\} \to G\backslash\{0\}$ is called a *Ring* if the $*$ is associative and distributes over the $+$ operation, meaning that for all x, y and $z \in G$:

$$x * (y + z) = x * y + x * z \text{ and } (y + z) * x = y * x + z * x$$

In addition to this, if for a Ring G it happens that for all x, $y \in G$, $x * y = y * x$, then the Ring is called *Commutative.* If the Ring $(G, +, *)$ happens to have an identity element for $*$, then it is called a *Unitary Ring.*

- Prove that $(\mathbb{Z}, +, *)$ in which $+$ is the standard addition of integers and $*$ is the standard multiplication is a commutative unitary ring.

- On the Ring $(\mathbb{Z}, +, *)$ define another multiplication by

$$a *' b = a + b - a * b.$$

Which properties of a Ring does $(\mathbb{Z}, +, *')$ have?

If in a commutative unitary Ring every element has a multiplicative inverse, the Ring is called a Field.

- Prove that $(\mathbb{Z}, +, *)$ is not a Field.
- Prove that $(\mathbb{Z}_7, +, *)$ where $+$ and $*$ are addition and multiplication modulo 7 respectively, is a Field.
- Prove that in a Filed, multiplication of two non-zero elements are non-zero.

Except for a Field, in some Rings it can happen that the multiplication of two non-zero elements is zero. These elements have a special name.

Definition An element a of a Ring $(G, +, *)$ is called a left (right) zero-divisor if there exists a non-zero element $g \in G$ such that $a*g = 0$ ($g*a = 0$).

- Prove that $(\mathbb{Z}, +, *)$, has no zero-divisors except 0.
- Are there any zero-divisors in the Ring $(\mathbb{Z}_7, +, *)$? What about $(\mathbb{Z}_8, +, *)$? $(\mathbb{Z}_p, +, *)$ where p is a prime number?

Please send your answers to “ayse.bilge@khas.edu.tr”.

References

[1] Hungerford, T. W. (1980). Algebra, volume 73 of. Graduate Texts in Mathematics.