$\sqrt{2}$ is not rational!

Between two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ one can always find a new rational number, for instance $\frac{a+c}{b+d}$ and we can continue this to find infinitely many rational numbers. However no matter how much we continue this process, there will always be some *gaps*, for instance the hypotenuse of a right triangle with both sides equal to 1 (which is a number equal to 2 when squared) can not be expressed as a fraction $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Can you prove this?

Theorem (The Integer Root Theorem - Gauss) The rational roots of the monic polynomial $x^n + a_n x^{n-1} + \cdots + a_0 = 0$ with $a_i \in \mathbb{Z}$ for all i, are integers. (Can you prove this?). Using this theorem construct a polynomial with integer coefficients and use it to show that $\sqrt{2}$ is not rational.

Please send your answers to "ayse.bilge@khas.edu.tr".