Diophantine Equations

Definition (Diophantine Equation) Any polynomial equation which usually involve two or more variables from the set of natural numbers (sometimes integers and even rationals are considered also), is called a *Diophantine Equation*.

- Consider the equation
$$a + b = c$$
where a, b, $c \in \mathbb{N}$. This equation, which is an instance of a *linear* Diophantine equation, have infinitely many solutions in Natural numbers. Find at least three (a, b, c) which satisfy this equation.

- Now consider the equation
$$a^2 + b^2 = c^2$$
where a, b, $c \in \mathbb{N}$. The solutions to this equation are called *Pythagorean triples*. You may know that $(3, 4, 5)$ is a Pythagorean triple since $3^2 + 4^2 = 5^2$. Find at least three other Pythagorean triples. Can you find all the solutions to this equation?

- Consider the equation
$$a^3 + b^3 = c^3$$
where a, b, $c \in \mathbb{N}$. Can you find any solutions? What about the more general form $a^n + b^n = c^n$ for $n \geq 3$?

The question of finding the solutions for the equation

$$a^n + b^n = c^n$$

for $n \geq 3$ is known as *Fermat's Last Theorem*. On the margin of his 'Diophantus's Arithmetica', Fermat had noted that he has an amazing proof for

this but the margin is too small for the proof to be fit in. The theorem was proved by Sir Andrew Wiles in 1994, which took more than 300 pages. Interestingly there is an open problem in Number Theory known as *The abc conjecture*, which if proved to be true, can imply Fermat's last theorem in a way that the proof can be fit on the margin!

Please send your answers to "ayse.bilge@khas.edu.tr".

References

[1] Hardy, G. H., & Wright, E. M. (1979). An introduction to the theory of numbers. Oxford university press.

[2] Nathanson, M. B. (2008). Elementary methods in number theory (Vol. 195). Springer Science & Business Media.